Two-stage image denoising by principal component analysis with local pixel grouping

2014.2.13

Hsin-Hui Chen
Outline

• Principal component analysis (PCA)
• Local pixel grouping (LPG)
• LPG-PCA
LPG-PCA
Denote by $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_m]^T$ an $m$-component vector variable

$$
\mathbf{x} = \begin{bmatrix}
x_1^1 & x_1^2 & \ldots & x_1^n \\
x_2^1 & x_2^2 & \ldots & x_2^n \\
\vdots & \vdots & \ddots & \vdots \\
x_m^1 & x_m^2 & \ldots & x_m^n 
\end{bmatrix}
$$
LPG-PCA

The $i$th row of sample matrix $\mathbf{X}$,

$$X_i = [x_i^1 \ x_i^2 \ \ldots \ x_i^n]$$

The mean value of $X_i$ is calculated as

$$\mu_i = \frac{1}{n} \sum_{j=1}^{n} X_i(j)$$

the sample vector $X_i$ is centralized as

$$\mathbf{X_i} = X_i - \mu_i = [\bar{x}_i^1 \ \bar{x}_i^2 \ \ldots \ \bar{x}_i^n]$$
LPG-PCA

The goal of PCA is to find an orthonormal transformation matrix $\mathbf{P}$ to de-correlate $\mathbf{X}$, i.e. $\mathbf{Y} = \mathbf{P}\mathbf{X}$ so that the co-variance matrix of $\mathbf{Y}$ is diagonal.
Finally, the co-variance matrix of the centralized dataset is calculated as

$$
\Omega = \frac{1}{n}XX^T
$$

Since the co-variance matrix $\Omega$ is symmetrical, it can be written as:

$$
\Omega = \Phi \Lambda \Phi^T
$$

where $\Phi = [\phi_1 \phi_2 \ldots \phi_m]$ is the $m \times m$ orthonormal eigenvector matrix and $\Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the diagonal eigenvalue matrix with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. 
LPG-PCA

The terms $\phi_1, \phi_2, \ldots, \phi_m$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvectors and eigenvalues of $\Omega$. By setting

$$P = \Phi^T$$

$\bar{X}$ can be decorrelated, i.e. $\bar{Y} = PX$ and $\Lambda = (1/n)\bar{Y}\bar{Y}^T$. 
LPG-PCA

An important property of PCA is that it fully de-correlates the original dataset $\mathbf{X}$. Generally speaking, the energy of a signal will concentrate on a small subset of the PCA transformed dataset, while the energy of noise will evenly spread over the whole dataset. Therefore, the signal and noise can be better distinguished in the PCA domain.
LPG-PCA

for an underlying pixel to be denoised, we set a $K \times K$ window centered on it and denote by $\mathbf{x} = [x_1 \ldots x_m]^T$, $m = K^2$, the vector containing all the components within the window.

Since the observed image is noise corrupted, we denote by

$$\mathbf{x}_v = \mathbf{x} + \mathbf{v}$$

the noisy vector of $\mathbf{x}$, where $\mathbf{x}_v = [x_1^v \ldots x_m^v]^T$, $\mathbf{v} = [v_1 \ldots v_m]^T$
LPG-PCA

Next we centralize dataset $X_b$.

$$\overline{X}_b = \overline{X} + V$$

by computing the covariance matrix of $\overline{X}$, denoted by $\Omega_x$, the PCA transformation matrix $P_x$ can be obtained.
LPG-PCA

However, the available dataset $X_v$ is noise corrupted so that $\Omega_{\bar{X}}$ cannot be directly computed. With the linear model (3.5), we have

$$\Omega_{\bar{X}_v} = \frac{1}{n} \bar{X}_v \bar{X}_v^T = \frac{1}{n} (\bar{X}X^T + \bar{X}V^T + V\bar{X}^T + VV^T)$$

Since $\bar{X}$ and $V$ are uncorrelated, items $\bar{X}V^T$ and $V\bar{X}^T$ will be nearly zero matrices and thus:

$$\Omega_{\bar{X}_v} \approx \frac{1}{n} (\bar{X}X^T + VV^T) = \Omega_{\bar{X}} + \Omega_v$$

where $\Omega_{\bar{X}} = (1/n) \bar{X}X^T$ and $\Omega_v = (1/n)VV^T$. 
LPG-PCA

we can decompose $\Omega_x$ as

$$
\Omega_x = \Phi_x \Lambda_x \Phi_x^T
$$

where $\Phi_x$ is the $m \times m$ orthonormal eigenvector matrix and $\Lambda_x$ is the diagonal eigenvalue matrix.
LPG-PCA

Since $\Phi_X$ is an orthonormal matrix, we can write $\Omega_v$ as

$$\Omega_v = (\sigma^2 I) \Phi_X \Phi_T^T = \Phi_X (\sigma^2 I) \Phi_T^T = \Phi_X \Omega_v \Phi_T^T$$

Thus we have

$$\Omega_{x_0} = \Omega_X + \Omega_v = \Phi_X \Lambda_X \Phi_T^T + \Phi_X (\sigma^2 I) \Phi_T^T$$

$$= \Phi_X (\Lambda_X + \sigma^2 I) \Phi_T^T = \Phi_X \Lambda_{x_0} \Phi_T^T$$
LPG-PCA

the orthonormal PCA transformation matrix for $\bar{X}$ is set as

$$P_{\bar{X}} = \Phi_{\bar{X}}^T$$

Applying $P_{\bar{X}}$ to dataset $\bar{X}_v$, we have

$$\bar{Y}_v = P_{\bar{X}}\bar{X}_v = P_{\bar{X}}\bar{X} + P_{\bar{X}}V = \bar{Y} + V_Y$$
LPG-PCA

Since $\bar{\mathbf{Y}}$ and noise $\mathbf{V}_Y$ are uncorrelated, we can easily derive that the covariance matrix of $\bar{\mathbf{Y}}_v$ is

$$\Omega_{\bar{\mathbf{Y}}_v} = \frac{1}{n} \bar{\mathbf{Y}}_v \bar{\mathbf{Y}}_v^T = \Omega_{\bar{\mathbf{Y}}} + \Omega_{\mathbf{V}_y}$$

where $\Omega_{\bar{\mathbf{Y}}} = \Lambda_{\bar{\mathbf{x}}}$ is the covariance matrix of decorrelated dataset $\bar{\mathbf{Y}}$ and $\Omega_{\mathbf{V}_y} = P_{\bar{\mathbf{x}}} \Omega_{\mathbf{V}_y} P_{\bar{\mathbf{x}}}^T$ is the covariance matrix of noise dataset $\mathbf{V}_Y$. 
LPG-PCA

Since $\overline{Y}_v$ is centralized, the LMMSE of $\overline{Y}_k$, i.e. the $k$th row of $\overline{Y}$, is obtained as

$$\hat{\overline{Y}}_k = w_k \cdot \overline{Y}_v$$

where the shrinkage coefficient

$$w_k = \frac{\Omega_{\overline{Y}}(k, k)}{\Omega_{\overline{Y}}(k, k) + \Omega_{yy}(k, k)}$$
LPG-PCA

In implementation we first calculate $\Omega_{\overline{Y}_v}$ from the available noisy dataset $\overline{Y}_v$ and then estimate $\Omega_{\overline{Y}}(k, k)$ by $\Omega_{\overline{Y}}(k, k) = \Omega_{\overline{Y}_v}(k, k) - \Omega_{\nu_y}(k, k)$.

In flat zones, it is often that $\Omega_{\overline{Y}_v}(k, k) - \Omega_{\nu_y}(k, k) \leq 0$, and then we set $\Omega_{\overline{Y}}(k, k) = 0$. In this case $w_k$ will be exactly 0 and all the noise in $\overline{Y}_v$ will be removed.
LPG-PCA

Denote by $\hat{\mathbf{Y}}$ the matrix of all $\overline{Y}_k$. By transforming $\hat{\mathbf{Y}}$ back to the time domain, we obtain the denoised result of $\overline{X}_b$ as

$$\hat{\mathbf{X}} = \mathbf{P}_\overline{X}^T \cdot \hat{\mathbf{Y}}$$
Adding the mean values $\mu_k$ back to $\bar{X}$ gives the denoised dataset $\hat{X}$. The estimation of the central block $\hat{x}_0$, denoted as $\hat{x}_0$, can then be extracted from $\hat{X}$ and finally the denoised result of the underlying central pixel can be extracted from $\hat{x}_0$.

Applying the above procedure to each pixel leads to the full denoised image of $I_o$. 
LPG-PCA

Denote by $\hat{l}$ the denoised version of noisy image $l_n$ in the first stage. We can write $\hat{l}$ as $\hat{l} = l + v_s$, where $v_s$ is the residual in the denoised image.
LPG-PCA

need to estimate the level of \( v_s \), denoted by \( \sigma_s = \sqrt{E[v_s^2]} \), and input it to the second stage of LPG-PCA denoising. Here we estimate \( \sigma_s \) based on the difference between \( \hat{I} \) and \( I_o \). Let

\[
\tilde{I} = I_o - \hat{I} = v - v_s
\]

We have:

\[
E[\tilde{I}^2] = E[v^2] + E[v_s^2] - 2E[v \cdot v_s] \\
= \sigma^2 + \sigma_s^2 - 2E[v \cdot v_s]
\]
LPG-PCA

We approximately view \( v_s \) as the smoothed version of noise \( v \), and it contains mainly the low frequency component of \( v \). Let \( \tilde{v} = v - v_s \) be their difference and \( \tilde{v} \) contains mainly the high frequency component of \( v \). There is \( E[v \cdot v_s] = E[\tilde{v} \cdot v_s] + E[v_s^2] \).
LPG-PCA

\[ E[v \cdot v_s] = E[\tilde{v} \cdot v_s] + E[v_s^2] \approx E[v_s^2] = \sigma_s^2 \]

\[ \sigma_s^2 \approx \sigma^2 - E[\tilde{I}^2] \]
LPG-PCA

In practice, $v_s$ will include not only the noise residual but also the estimation error of noiseless image $I$. Therefore, in implementation we let

$$\sigma_s = c_s \cdot \sqrt{\sigma^2 - E[\tilde{I}^2]}$$

where $c_s < 1$ is a constant. We experimentally found that setting $c_s$ around 0.35 can lead to satisfying denoising results for most of the testing images.
Q&A