Spatial-Slepian Transform on the Sphere

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Outline

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- Signals on the $\mathbb{SO}(3)$ Rotation Group
- Spatial-Spectral Concentration on the Sphere
- Spatial-Slepian Transform
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Signals on the Sphere

In the surface of a unit sphere is defined as $\mathbb{S}^2$.

- Orthonormal basis functions called spherical harmonics

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)e^{im\phi}.$$

Any signal $f \in L^2(\mathbb{S}^2)$ can be expanded as

$$f(\widehat{\boldsymbol{x}}) = \sum_{l,m}^{\infty}(f)_l^m Y_n^m(\widehat{\boldsymbol{x}})$$

where

$$(f)_l^m = \langle f, Y_l^m \rangle_{\mathbb{S}^2}$$

Signal Rotation on the Sphere

- Rotation matrices

$$\mathbf{R} \equiv \mathbf{R}_z(\varphi)\mathbf{R}_y(\vartheta)\mathbf{R}_z(\omega)$$

where $\mathbf{R}_y(\vartheta)$ rotate by angles ϑ around *y*-axis.

- Rotation operator of Euler angles $\rho = (\varphi, \vartheta, \omega)$ is

$$\mathcal{D}_\rho \equiv \mathcal{D}(\varphi, \vartheta, \omega)$$

- Signal Rotation on the Sphere

$$(\mathcal{D}_\rho f)(\hat{\boldsymbol{x}}) = f(\mathbf{R}^{-1}\hat{\boldsymbol{x}}).$$

Signal Rotation on the Sphere

- Signal rotation on the sphere

$$\left(\mathcal{D}_\rho f\right)(\widehat{\boldsymbol{x}}) = f(\mathbf{R}^{-1}\widehat{\boldsymbol{x}}) = \sum_{l,m}^{\infty} \underbrace{\left(\sum_{m'=-l}^{l} \mathcal{D}_{m,m'}^{l}(\rho)\,(f)_l^{m'}\right)}_{Spectral\ coefficients} Y_l^m(\widehat{\boldsymbol{x}})$$

where $\mathcal{D}_{m,m'}^{l}$ is the Wigner *D*-function.

$$\mathcal{D}_{m,m'}^{l}(\rho) = e^{-im\varphi} d_{m,m'}^{l}(\vartheta) e^{-im'\omega}$$

and $d_{m,m'}^{l}$ is the Wigner's (small) *d*-function.

Signals on the $\mathbb{SO}(3)$ Rotation Group

All rotations by $\rho = (\varphi, \vartheta, \omega)$ is called the $\underbrace{\textbf{Special}}_{det=1}$ **Orthogonal** $\underbrace{\textbf{group}}_{\text{closure, inverse}}$ $\mathbb{SO}(3)$.

Wigner *D*-functions form the basis functions on the $\mathbb{SO}(3)$, since the orthogonality

$$\left\langle \mathcal{D}^{l}_{m,\,m'}, \mathcal{D}^{p}_{m,\,m'} \right\rangle_{\mathbb{SO}(3)} = \left(\frac{8\pi^2}{2l+1}\right) \delta_{l,p} \delta_{m,q} \delta_{m',q'} \,.$$

Any signal $v \in L^2\big(\mathbb{SO}(3)\big)$ can be expanded as

$$v(\rho) = \sum_{l,m}^{\infty} (v)^{l}_{m,\,m'} \overline{\mathcal{D}^{l}_{m,\,m'}(\rho)}$$

where

$$(v)^{l}_{m,\,m'} = \left(\frac{8\pi^2}{2l+1}\right) \left\langle v, \overline{\mathcal{D}^{l}_{m,\,m'}} \right\rangle_{\mathbb{SO}(3)}$$

Spatial-Spectral Concentration on the Sphere

A bandlimited $(l = L_g)$ signal g in $R \subset \mathbb{S}^2$.

- Spatial energy concentration

$$\lambda = \frac{\|g\|_R^2}{\|g\|_{\mathbb{S}^2}^2} = \frac{\sum_{l,m}^{L_g-1} \sum_{p,q}^{L_g-1} \overline{(g)_l^m} (g)_p^q \, K_{lm,pq}}{\sum_{l,m}^{L_g-1} \left|(g)_l^m\right|^2} = \frac{\mathbf{g}^{\mathrm{H}} \mathbf{K} \mathbf{g}}{\mathbf{g}^{\mathrm{H}} \mathbf{g}}$$

where

$$K_{lm,pq} = \int_R \overline{Y_l^m(\hat{\boldsymbol{x}})} Y_p^q(\hat{\boldsymbol{x}}) \, ds$$

is called spherical harmonics double product.

$\mathbf{K}$ is Hermitian and positive definite, the eigenvalues λ are real and eigenvectors $\mathbf{g}$ are orthogonal.

Slepian functions

- Eigenvalues

$$1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{L_g^2} > 0.$$

- Slepian functions (eigenvectors)

$$g_1(\hat{\boldsymbol{x}}), g_2(\hat{\boldsymbol{x}}), \cdots, g_{L_g^2}(\hat{\boldsymbol{x}}).$$

- Any signal $f \in BL_{L_g}$ can be expanded as

$$f(\hat{\boldsymbol{x}}) = \sum_{\alpha=1}^{L_g^2} (f)_\alpha g_\alpha(\hat{\boldsymbol{x}})$$

where

$$(f)_\alpha = \langle h, g_\alpha \rangle_{\mathbb{S}^2} = \mathbf{g}_\alpha^{\mathrm{H}} \mathbf{h}.$$

Clustering Behavior of the Eigenvalues

The corresponding eigenvalues $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{L_g^2} > 0$.

Most of eigenvalues are either nearly 1 or nearly 0.

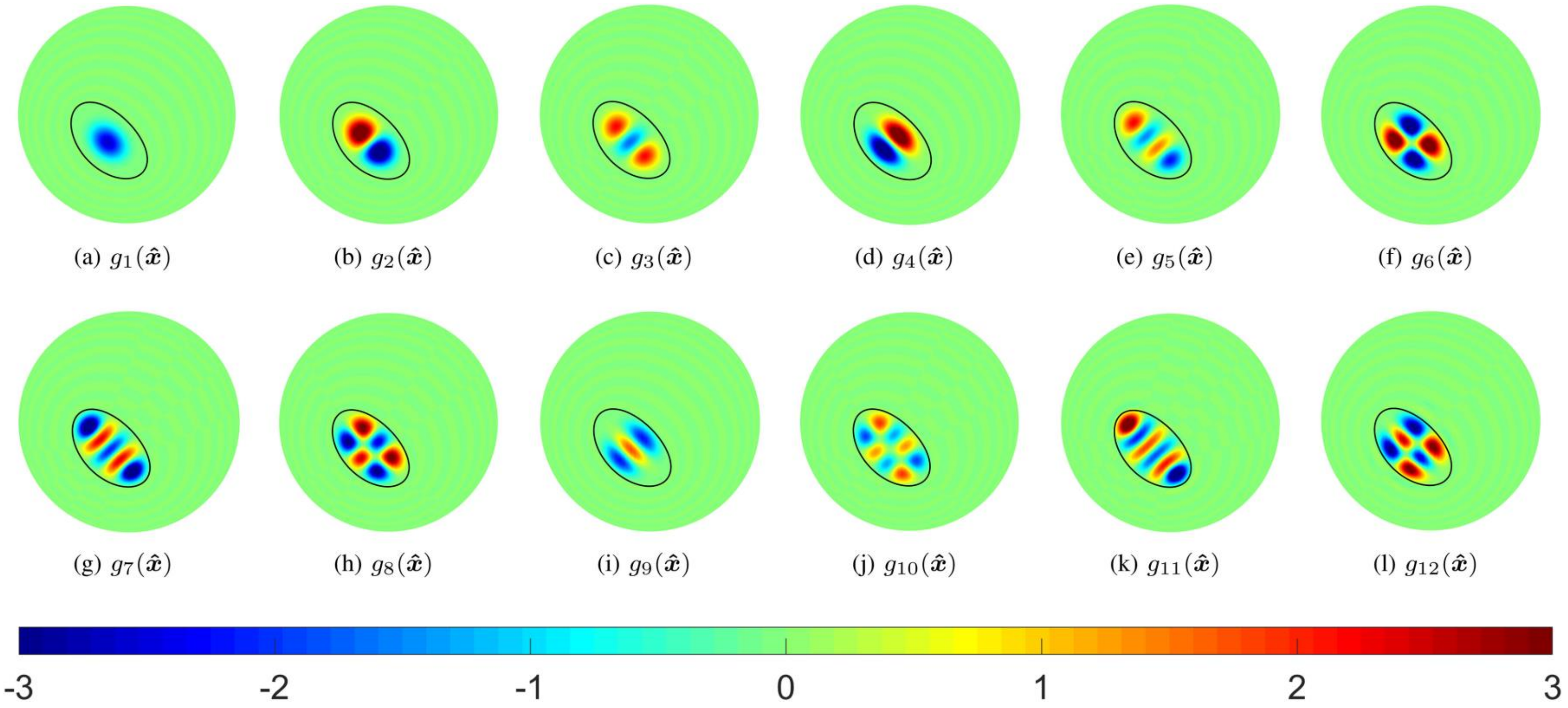
- Spherical Shannon number (sum of eigenvalues)

$$N_R \triangleq \sum_{\alpha=1}^{L_g^2} \lambda_\alpha = trace(\mathbf{K}) = \frac{A_R}{4\pi} L_g^2$$

where $A_R \triangleq \|1\|_R$ is the surface area of the spatial region R.

The first N_R concentrated Slepian functions form a localized basis set of bandlimited signals in the spatial region R.

Example: Slepian functions



A. Aslam, and Z. Khalid, "Spatial-Slepian Transform on the Sphere," *IEEE Trans. Signal Process.*, vol. 69, pp. 4474-4485, 2021.

Spatial-Slepian Transform

Well-optimally concentrated Slepian functions $g_\alpha(\hat{\boldsymbol{x}}), \alpha = 1, 2, \cdots, N_R$.

- Spatial-Slepian Transform

$$F_{g_\alpha}(\rho) \triangleq \langle f, \mathcal{D}_\rho g_\alpha \rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} f(\hat{\boldsymbol{x}}) \overline{\mathcal{D}_\rho g_\alpha(\hat{\boldsymbol{x}})}\, ds(\hat{\boldsymbol{x}})$$

$$= \sum_{l,m}^{min\{L_f-1, L_g-1\}} (f)_l^m \overline{(g_\alpha)_l^{m'} \mathcal{D}_{m,m'}^l(\rho)}$$

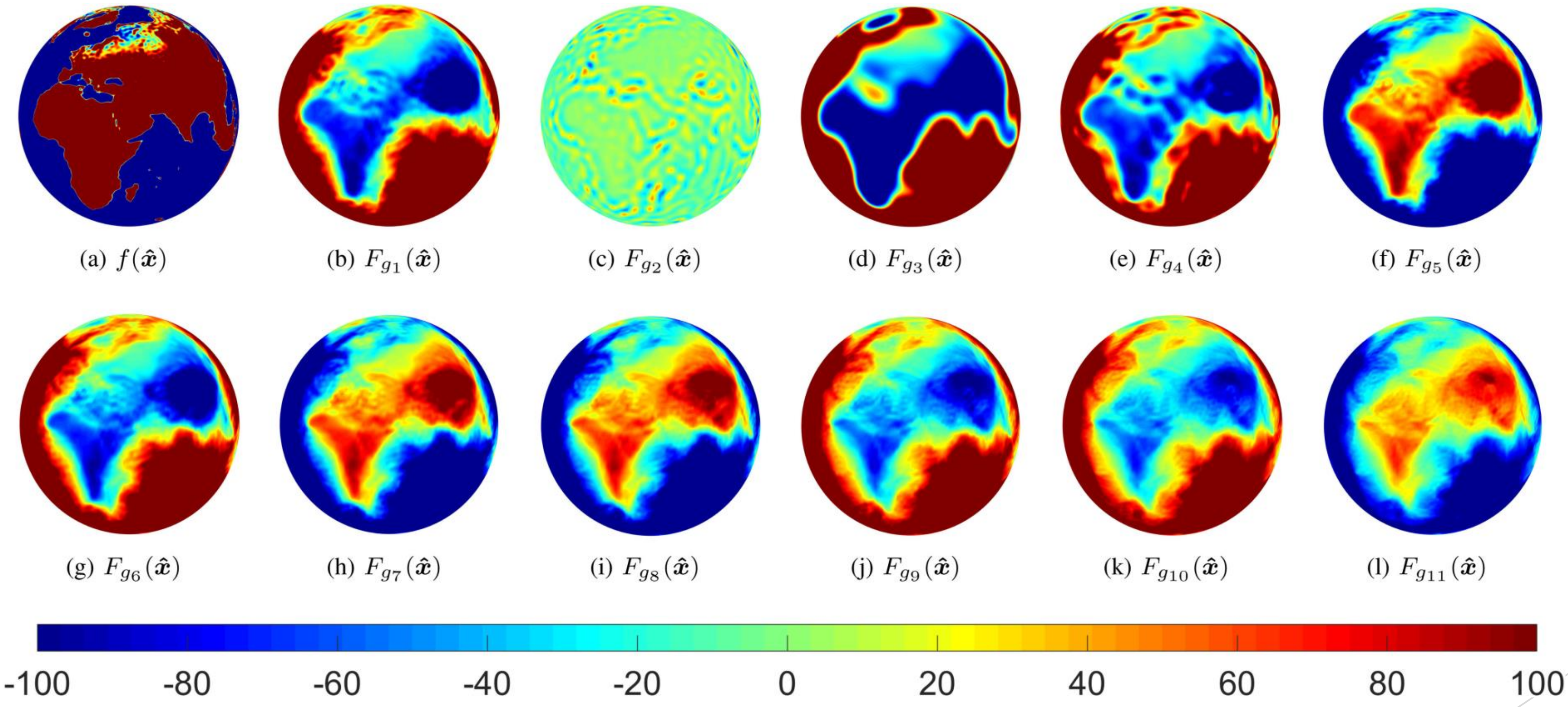
Inverse Spatial-Slepian Transform

- Inverse Spatial-Slepian Transform

$$\left(F_{g_\alpha}\right)^{l}_{m,m'} \triangleq \left(\frac{2l+1}{8\pi^2}\right)\left\langle F_{g_\alpha}, \overline{\mathcal{D}^{l}_{m,m'}}\right\rangle_{\mathbb{SO}(3)} = (f)^{m}_{l}\overline{(g_\alpha)^{m'}_{l}}$$

$$(f)^{m}_{l} = \left(\frac{2l+1}{8\pi^2}\right)\frac{\left\langle F_{g_\alpha}, \overline{\mathcal{D}^{l}_{m,m'}}\right\rangle_{\mathbb{SO}(3)}}{\overline{(g_\alpha)^{m'}_{l}}}$$

Example: Spatial-Slepian Transform



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Complexity

$$F_{g_\alpha}(\rho) = \sum_{m,n,k=-(L_f-1)}^{L_f-1} C_{m,n,k}^{\alpha}\, e^{i(m\varphi+n\vartheta+k\omega)}$$

- $O\left(L_f^3 log_2 L_f\right)$

where

$$C_{m,n,k}^{\alpha} = i^{m-n} \sum_{l=max\{|m|,|n|,|k|\}}^{L_f-1} (f)_l^m\, \overline{(g_\alpha)_l^n}\, d_{k,m}^l(\pi/2)\, d_{k,n}^l(\pi/2)$$

- $O\left(L_f^4\right)$

Conclusion

New Approach provide

- Bandlimited and spatially limited Slepian functions on the sphere.
- The first N_R (spherical Shannon number) concentrated Slepian functions.

Reference

1. A. Aslam, and Z. Khalid, “Spatial-Slepian Transform on the Sphere,” *IEEE Trans. Signal Process.*, vol. 69, pp. 4474-4485, 2021.