Integer Transform

- What Integer Transform
  - Definition and Property
  - Verifying Fixed-point Number
  - Integer Transform
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Fixed-point number

$$y = \sum_k a_k 2^k \; where \; a_k \in \{0,1\}, k \in \mathbb{Z}$$

For example:

$$59.125 = 2^5 + 2^4 + 2^3 + 2^1 + 2^0 + 2^{-3}$$

k	**5**	**4**	**3**	**2**	**1**	**0**	**-1**	**-2**	**-3**
a_k	**1**	**1**	**1**	**0**	**1**	**1**	**0**	**0**	**1**

Radix point (between k = 0 and k = -1)

Verifying fixed-point number

$$59.125 = 2^5 + 2^4 + 2^3 + 2^1 + 2^0 + 2^{-3}$$

Radix point (between k = 0 and k = -1)

k	5	4	3	2	1	0	-1	-2	-3
a_k	1	1	1	0	1	1	0	0	1

k	8	7	6	5	4	3	2	1	0
a_k	1	1	1	0	1	1	0	0	1

Radix point (after k = 0)

$$473 = 2^3 \times 59.125 = 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^0$$

$$\frac{\sum_k a_k 2^k}{2^{k_0}}, \textit{ where } k \in \{0,1,2,\ldots\}, \textit{ for some } k_0$$

Integer transform

- The entries of the transform matrix $\boldsymbol{A_{ij}}$ can be represented by fixed-point number.
- Property:
  - If the entries of the input $\boldsymbol{v}_j = \sum_k a_{jk} 2^k \; for\, some\, a_{jk}$, then $\boldsymbol{Av} = \sum_k a_{jk} 2^k \; for\, some\, a_{jk}$

$$\begin{bmatrix} \sum_k a_k^{11} 2^k & \sum_k a_k^{12} 2^k \\ \sum_k a_k^{21} 2^k & \sum_k a_k^{22} 2^k \end{bmatrix} \begin{bmatrix} \sum_k x_k^{11} 2^k \\ \sum_k x_k^{21} 2^k \end{bmatrix} = \begin{bmatrix} \sum_k y_k^{11} 2^k \\ \sum_k y_k^{21} 2^k \end{bmatrix}$$

Goals to meet

I. If x is fixed-point number, that $y = A x$ is also fixed-point number

II. Reversibility $A^{-1} A x = x$

III. High accuracy

Some Problems of Integer Transform

$$If\ A = 10, A^{-1} = \frac{1}{10}!$$

$$If\ A = \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{8}{3} \end{bmatrix}!$$

$$det(A) = 2^k \wedge A\, is\, integer\ transform\ ,$$

$$\Leftrightarrow det(A^{-1}) = 2^p \wedge A^{-1}\, is\, also\, integer\ transform.$$

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Prototype Method

$$\begin{pmatrix} 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 & 0.3536 \\ 0.4904 & 0.4157 & 0.2778 & 0.0975 & -0.0975 & -0.2778 & -0.4157 & -0.4904 \\ 0.4619 & 0.1913 & -0.1913 & -0.4619 & -0.4619 & -0.1913 & 0.1913 & 0.4619 \\ 0.4157 & -0.0975 & -0.4904 & -0.2778 & 0.2778 & 0.4904 & 0.0975 & -0.4157 \\ 0.3536 & -0.3536 & -0.3536 & 0.3536 & 0.3536 & -0.3536 & -0.3536 & 0.3536 \\ 0.2778 & -0.4904 & 0.0975 & 0.4157 & -0.4157 & -0.0975 & 0.4904 & -0.2778 \\ 0.1913 & -0.4619 & 0.4619 & -0.1913 & -0.1913 & 0.4619 & -0.4619 & 0.1913 \\ 0.0975 & -0.2778 & 0.4157 & -0.4904 & 0.4904 & -0.4157 & 0.2778 & -0.0975 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & -d & -c & -b & -a \\ e & f & -f & -e & -e & -f & f & e \\ b & -d & -a & -c & c & a & d & -b \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ c & -a & d & b & -b & -d & a & -c \\ f & -e & e & -f & -f & e & -e & f \\ d & -c & b & -a & a & -b & c & -d \end{pmatrix}$$

i							
1	2	3	4	5	6	7	j
*3	*2	*3	*2	*3	*2	*3	0
	*3	*1	*3	*1	*3	*4	1
		*3	*2	*3	*4	*3	2
			*3	*4	*3	*1	3
				*3	*2	*3	4
					*3	*1	5
						*3	6

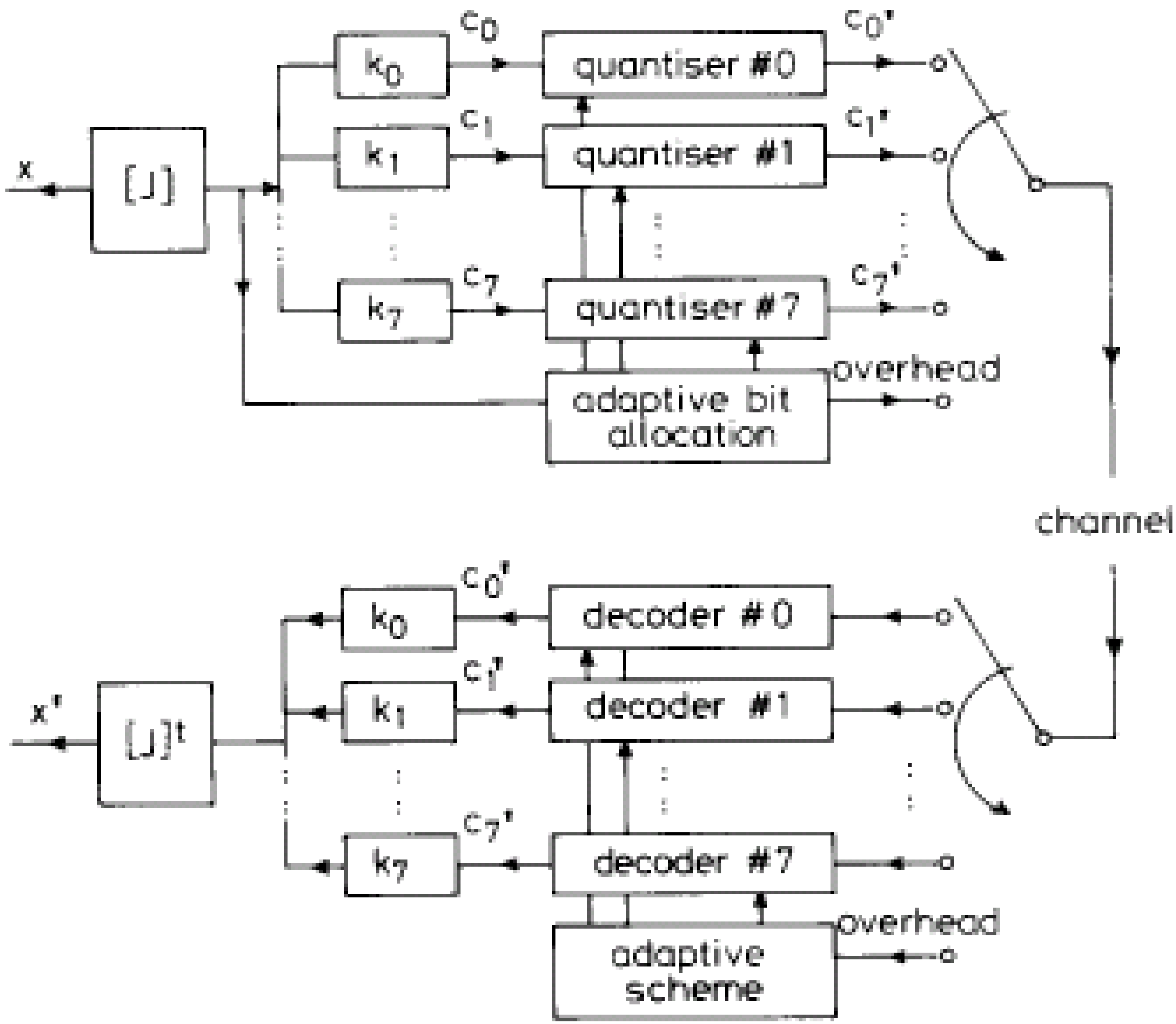
*1 if $b = a \cdot c + b \cdot d + c \cdot d$

*2 must be orthogonal due to the 3rd dyadic symmetry

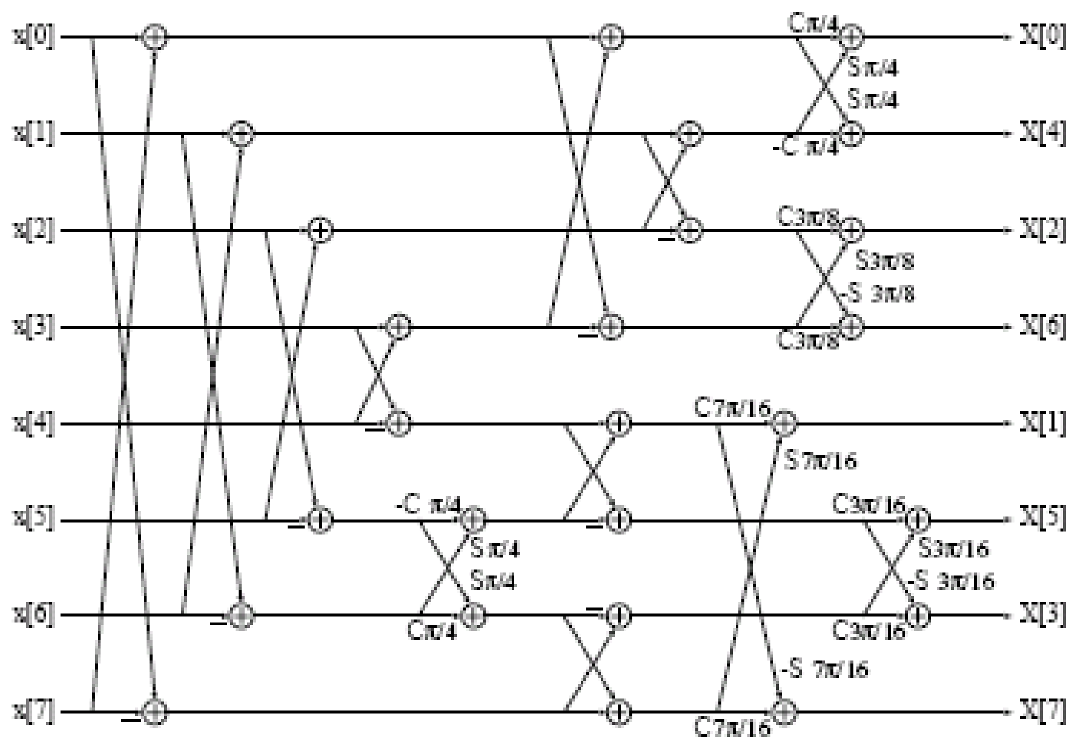
*3 must be orthogonal due to the 7th dyadic symmetry

*4 must be orthogonal as their dot product equals zero

Implementation



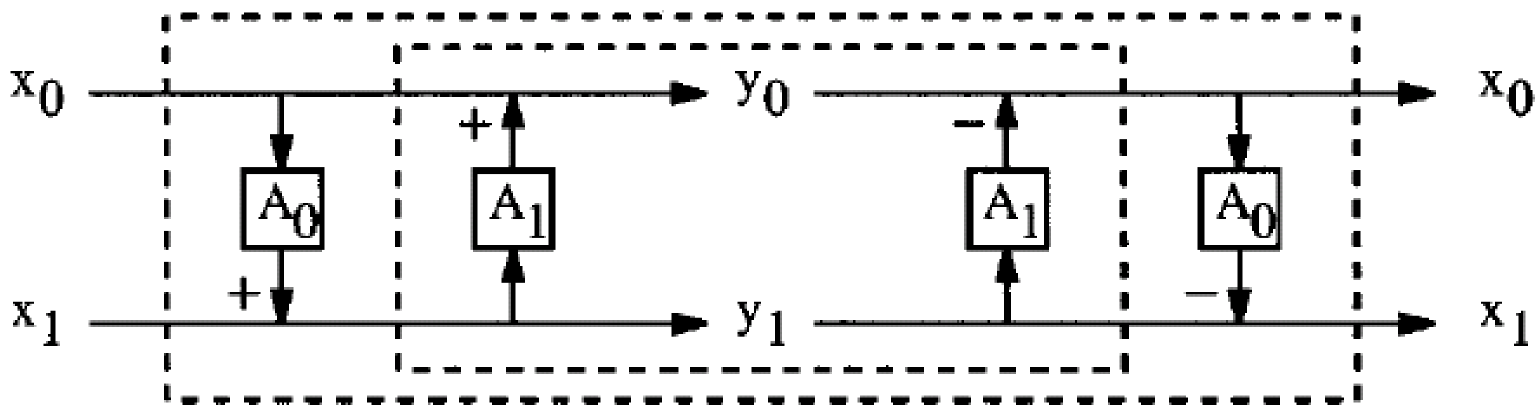
Lifting Scheme



Lifting Scheme

$$B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$

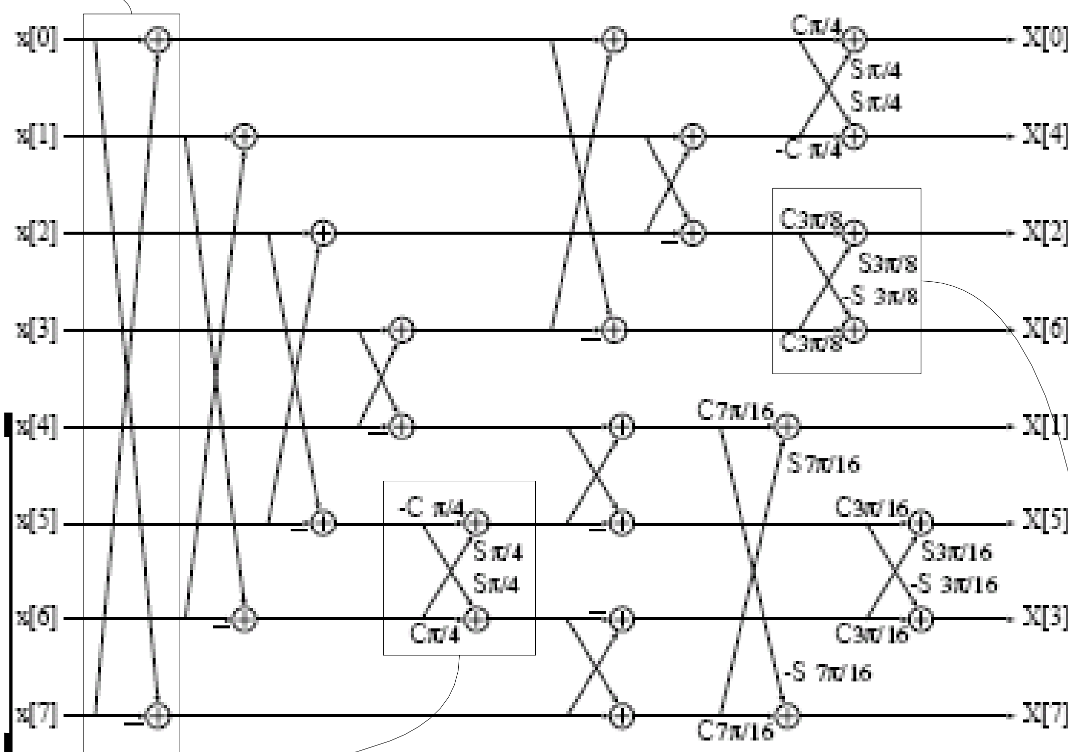


$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & A_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ A_0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \qquad \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -A_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -A_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Lifting Scheme

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$



$$\begin{bmatrix} -\cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$$

$$\begin{bmatrix} \cos\left(\frac{3\pi}{8}\right) & \sin\left(\frac{3\pi}{8}\right) \\ -\sin\left(\frac{3\pi}{8}\right) & \cos\left(\frac{3\pi}{8}\right) \end{bmatrix}$$

Lifting Scheme

Case 1

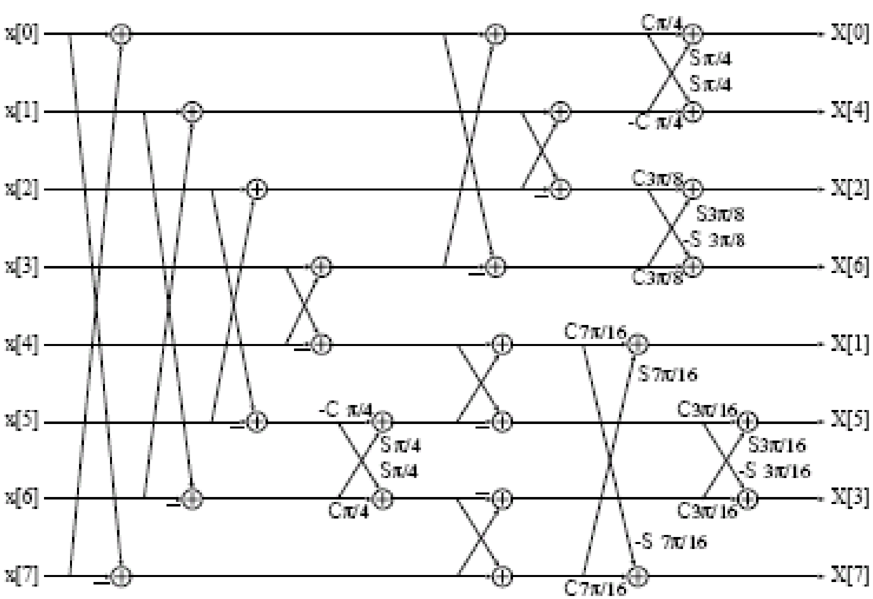
$$T_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{\cos(\theta)-1}{\sin(\theta)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin(\theta) & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{\cos(\theta)-1}{\sin(\theta)} \\ 0 & 1 \end{bmatrix}$$

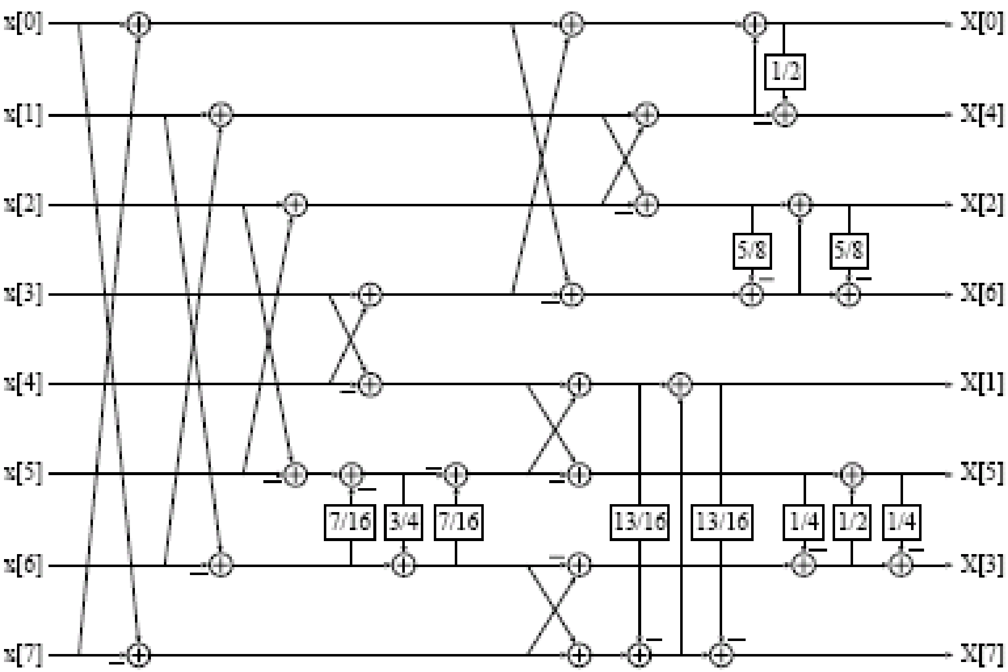
Case 2

$$T_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

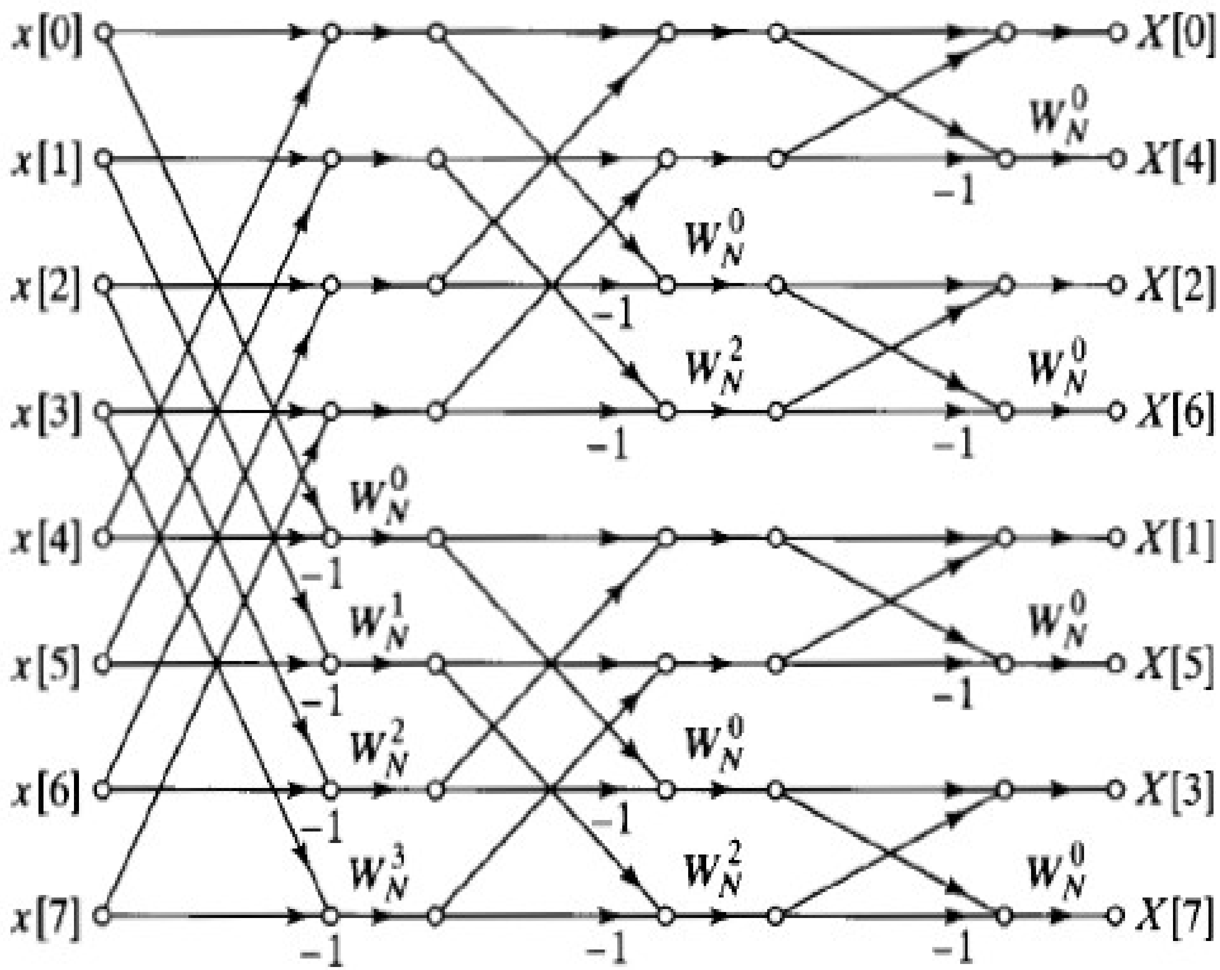
$$= \begin{bmatrix} 1 & \dfrac{\cos(\theta)-1}{\sin(\theta)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin(\theta) & 1 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{-(\cos(\theta)-1)}{\sin(\theta)} \\ 0 & -1 \end{bmatrix}$$

Lifting Scheme





Lifting Scheme

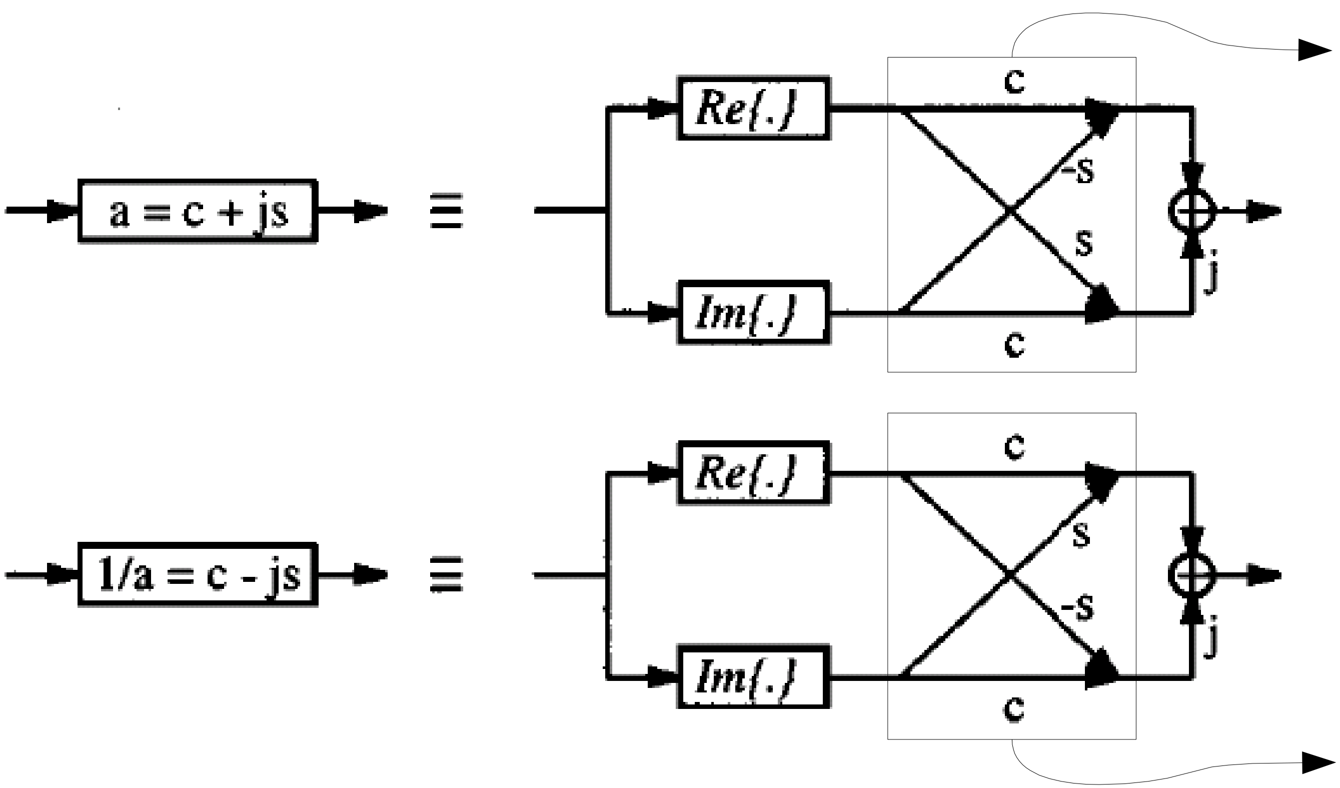


Lifting Scheme

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

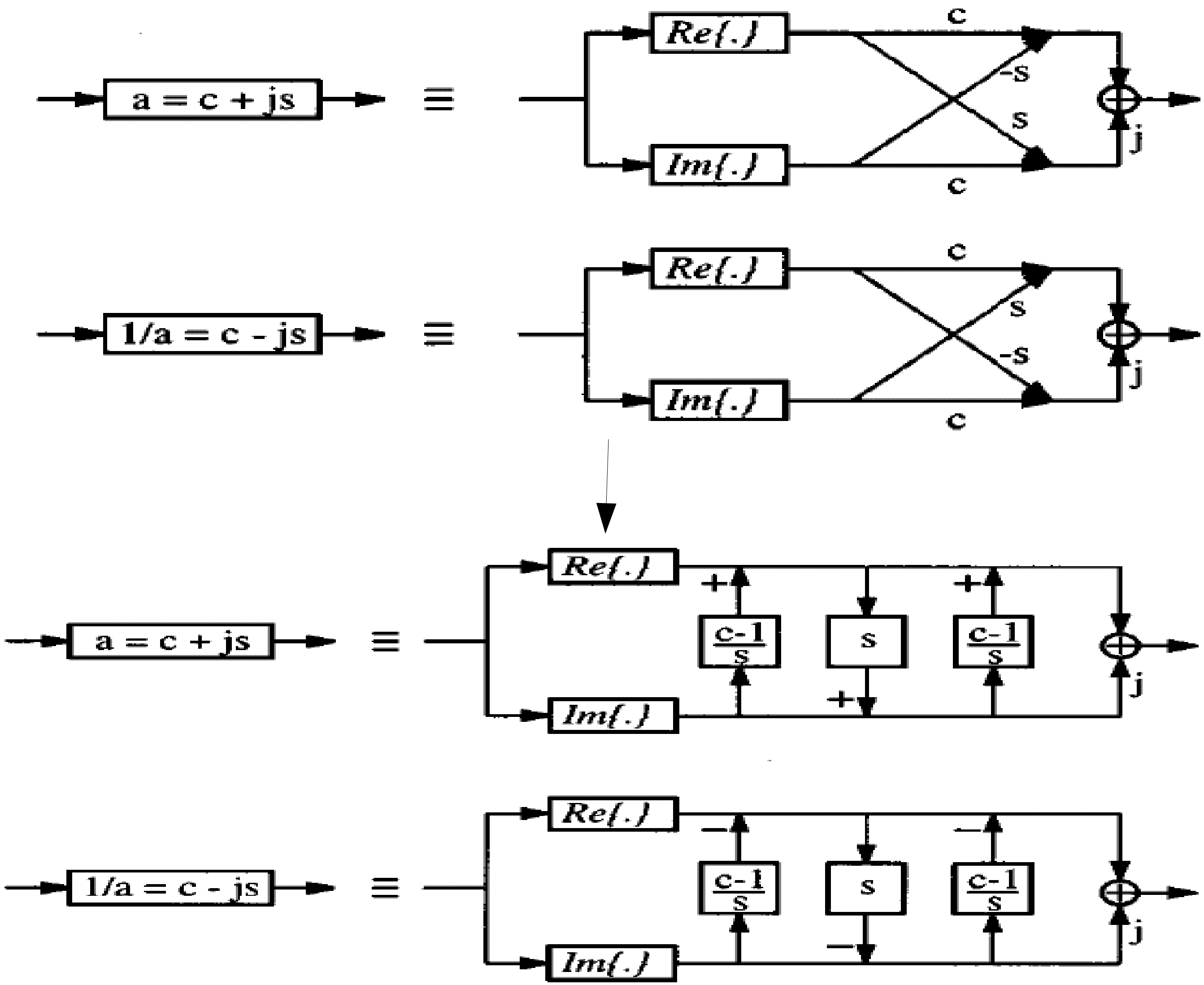
$$(a+bi)\times(c+di)=(ac-bd)+(ad+bc)i$$

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$



$$\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}$$

Lifting Scheme



Triangular Scheme

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & c & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & c & 1 & -b & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & ac-b & -c & 1 \end{array}\right]$$

Triangular Scheme

$$Assume\ |(det(A))| = 1, choose\ permutation\ P \wedge Q\ such\ that\ R = PAQ$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & s_{12} & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & K \end{bmatrix} = T_1 T_2 T_3$$

$$K = det(A)$$

Triangular Scheme

Algorithm I

$$\overset{A}{\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{bmatrix}} \overset{S_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & b_1 & c_1 \\ j_1 & e_1 & f_1 \\ k_1 & h_1 & i_1 \end{bmatrix} \qquad s_1 = \frac{1-a_1}{c_1}$$

$$\overset{L_1}{\begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & b_1 & c_1 \\ j_1 & e_1 & f_1 \\ k_1 & h_1 & i_1 \end{bmatrix} = \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & m_1 & n_1 \\ 0 & o_1 & p_1 \end{bmatrix} \qquad l_1 = -j_{1,} l_2 = -k_1$$

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & m_1 & n_1 \\ 0 & o_1 & p_1 \end{bmatrix} \overset{S_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_2 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & n_1 \\ 0 & q_1 & p_1 \end{bmatrix}$$

$$\overset{L_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_3 & 1 \end{bmatrix}} \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & n_1 \\ 0 & q_1 & p_1 \end{bmatrix} = \overset{U}{\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & n_1 \\ 0 & 0 & r_1 \end{bmatrix}}$$

$$L_2 L_1 A S_1 S_2 = U$$

$$Let\ L^{-1} = L_2 L_1 = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ l_2 + l_1 l_3 & l_3 & 1 \end{bmatrix}$$

$$S^{-1} = S_1 S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & s_2 & 1 \end{bmatrix}$$

$$Then\ A = LUS$$

Triangular Scheme

Algorithm II

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & 1 & 0 \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

$$a_{21}u_{12} + a_{22} = 1 \qquad a_{21}u_{13} + a_{22}u_{23} + a_{23} = 0$$

$$a_{31}u_{13} + a_{32}u_{23} + a_{33} = 1$$

$$\overset{S}{\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ h_{21} & 1 & 0 \\ h_{31} & h_{32} & 1 \end{bmatrix}} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & 1 & 0 \\ h_{31} & h_{32} & 1 \end{bmatrix}$$

$$s_{13} = h_{13}$$

$$s_{12} + s_{13}h_{32} = h_{12}$$

$$s_{11} + s_{12}h_{21} + s_{13}h_{31} = h_{11}$$

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} = \overset{S}{\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ h_{21} & 1 & 0 \\ h_{31} & h_{32} & 1 \end{bmatrix}} \overset{U^{-1}}{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}^{-1}}$$

Rounding

$$y = \frac{1}{4}x$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y_1 = \frac{1}{16}x_1 + \frac{1}{2}x_2$$

$$If\ Q(x+a) = x + Q(a)\ \forall\ x \in \mathbb{Z}\ ,\ we\ call\ Q\ is\ integer-shift\ invariant$$

Floor, Ceiling

Rounding

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Forward Transform

$$y_1 = Q(x_1 + u_{12} x_2 + u_{13} x_3) = x_1 + Q(u_{12} x_2 + u_{13} x_3)$$
$$y_2 = Q(x_2 + u_{23} x_3) = x_2 + Q(u_{23} x_3)$$
$$y_3 = Q(x_3) = x_3$$

Inverse Transform

$$x_3 = y_3$$
$$x_2 = y_2 - Q(u_{23} x_3)$$
$$x_1 = y_1 - Q(u_{12} x_2 + u_{13} x_3)$$

Error