Jump Plus AM-FM Mode Decomposition

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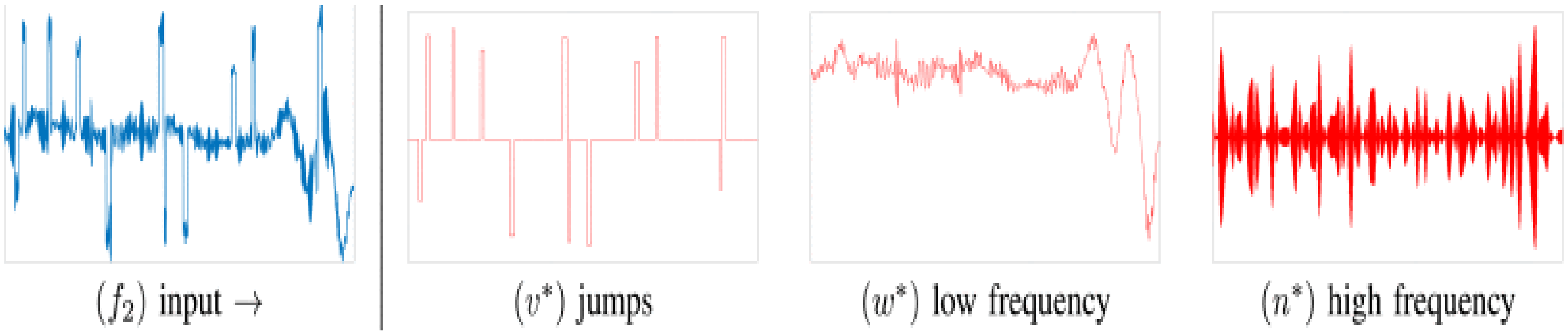
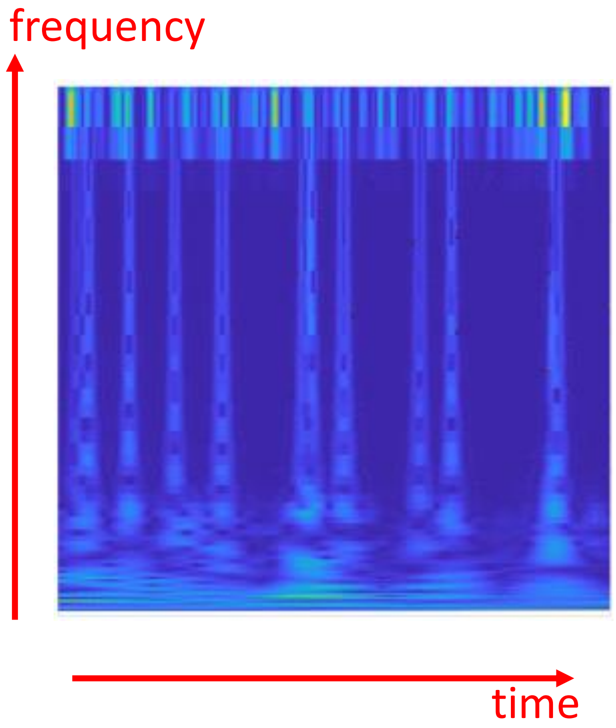
Introduction

- Such jumps, spikes and sudden drops can impact the physical meaningfulness of a signal decomposition [1].
- The discontinuities are wide-band components on the spectrum.





[1] A. Cicone, M. Huska, S. -H. Kang and S. Morigi, "JOT: A Variational Signal Decomposition Into Jump, Oscillation and Trend," in IEEE Transactions on Signal Processing, vol. 70, pp. 772-784, 2022.

Jump and oscillatory (AM-FM) modes

- Jump Plus AM-FM Mode Decomposition (JMD) [2] algorithm is proposed that jointly decomposes a nonstationary signal into jump and oscillatory (AM-FM) modes.
- The signal model:

$$f(t) = \sum_{k=1}^{K} u_k(t) + v(t) + n(t)$$

- $u_k(t)$: The K number of oscillatory or AM-FM components (or mode).
- $v(t)$: The jump component in the input data.
- $n(t)$: Noise component.

[2] M. Nazari, A. R. Korshøj and N. u. Rehman, "Jump Plus AM-FM Mode Decomposition," in IEEE Transactions on Signal Processing, vol. 73, pp. 1081-1093, 2025.

Oscillatory Components

- The Variational Mode Decomposition (VMD) algorithm [3] that aims to minimize the bandwidth of the AM-FM components:

$$J_1 = \sum_k \left\| \partial_t \left[u_k^+(t) e^{-j\omega_k t} \right] \right\|_2^2$$

- ∂_t: The partial derivative with respect to time.
- $u_k^+(t) = u_k(t) + j\mathcal{H}\{u_k(t)\}$: The analytic signal of the k-th mode.
- ω_k: The center frequency of each mode.

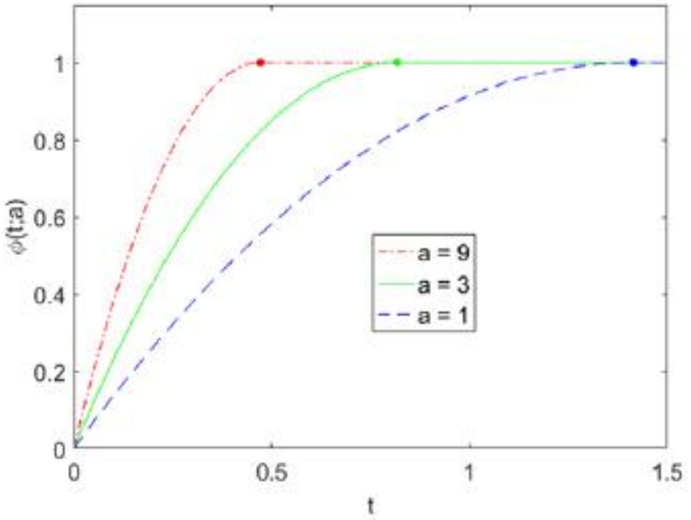
[3] K. Dragomiretskiy and D. Zosso, "Variational Mode Decomposition," in IEEE Transactions on Signal Processing, vol. 62, no. 3, pp. 531-544, Feb.1, 2014.

Jump Component

- This penalty term penalizes the derivative of the jump component to promote a piecewise-constant component:

$$J_2 = \int_0^{\infty} \phi(|\partial_t v(t)|; b) dt$$

- $\partial_t v(t)$: the first derivative of the jump component.
- $\phi(\cdot; a): [0, +\infty) \to [0,1]$: The non-convex sparsity-promoting penalty function is defined as a piecewise-quadratic function [4].



$$\phi(t; a) = \begin{cases} -\frac{a}{2} t^2 + \sqrt{2a} t, t \in [0, \sqrt{2/a}) \\ 1 \qquad\qquad\quad , t \in [\sqrt{2/a} + \infty) \end{cases}$$

$$\phi''(t; a) = \begin{cases} -a, t \in [0, \sqrt{2/a}) \\ 0 \quad , t \in [\sqrt{2/a} + \infty) \end{cases}$$

[4] M. Huska, A. Lanza, S. Morigi and I. Selesnick, "A convex-nonconvex variational method for the additive decomposition of functions on surfaces," in Inverse Problems, vol. 35, no. 12, pp. 124008, 2019.

Joint Formulation

- The formulation includes α and β for balancing the first and second terms:

$$\underset{\{u_k\},\{\omega_k\},v,x}{\text{argmin}} \quad \mathcal{J}(\{u_k\},\{\omega_k\},v,x)$$
$$\text{subject to } x = \partial_t v$$

where $\mathcal{J}(\{u_k\},\{\omega_k\},v,x) = \alpha \sum_k \left\| \partial_t \left[u_{k+}(t) e^{-j\omega_k t} \right] \right\|_2^2 + \beta \int_0^\infty \phi(|x(t)|;b)dt + \left\| f(t) - \left(\sum_{k=1}^{K} u_k(t) + v(t) \right) \right\|_2^2$

- The above formulation is solved using the alternate direction method of multipliers (ADMM).

Augmented Lagrangian

- The augmented Lagrangian:

$$\mathcal{L}(\{u_k\},\{\omega_k\},v,x,\rho)$$
$$= \alpha \sum_k \left\| \partial_t \left[u_{k+}(t) e^{-j\omega_k t} \right] \right\|_2^2 + \beta \int_0^\infty \phi(|x(t)|;b)dt$$
$$- \ \langle \rho(t), x(t) - \partial_t v(t) \rangle + \frac{\gamma}{2} \| x(t) - \partial_t v(t) \|_2^2$$
$$+ \left\| f(t) - \left(\sum_{k=1}^{K} u_k(t) + v(t) \right) \right\|_2^2$$

- $\rho(t)$: The dual variable.
- γ: The penalty scalar parameter.

Optimization Solution using ADMM - $\hat{u}_k^{(i+1)}, \omega_k^{(i+1)}$

- $\mathcal{L}(\{u_k\},\{\omega_k\}) = \alpha \sum_k \left\| \partial_t \left[u_{k+}(t) e^{-j\omega_k t} \right] \right\|_2^2 + \left\| f(t) - \left(\sum_{k=1}^K u_k(t) + v(t) \right) \right\|_2^2$
- Solve in the Fourier domain [3].
- $\hat{u}_k^{(i+1)}(\omega) = \frac{\hat{f}(\omega) - \sum_{l \neq k} \hat{u}_l^{(i)}(\omega) - \hat{v}^{(i)}(\omega)}{1 + 2\alpha \left(\omega - \omega_k^{(i)} \right)^2}$
- $\omega_k^{(i+1)} = \frac{\int_0^\infty \omega \left| \hat{u}_k^{(i+1)}(\omega) \right|^2 d\omega}{\int_0^\infty \left| \hat{u}_k^{(i+1)}(\omega) \right|^2 d\omega}$

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Optimization Solution using ADMM - $v^{(i+1)}$

$$
\begin{aligned}
&\mathcal{L}(v, x, \rho) \\
&= \beta \int_0^\infty \phi(|x(t)|; b)dt - \langle \rho(t), x(t) - \partial_t v(t) \rangle + \frac{\gamma}{2} \|x(t) - \partial_t v(t)\|_2^2 + \left\| f(t) - \left(\sum_{k=1}^{K} u_k(t) + v(t) \right) \right\|_2^2 \\
&= \beta \sum_{j=1}^{N} \phi(|x_j|; b) - \langle \rho, x - Dv \rangle + \frac{\gamma}{2} \|x - Dv\|_2^2 + \left\| f - \left(v + \sum_k u_k \right) \right\|_2^2
\end{aligned}
$$

(Assume that we deal with a finite signal.)

where D is defined as the first-order derivative matrix.

- $v^{(i+1)} = (\gamma D^T D + 2I)^{-1} \left(D^T \rho^{(i)} + \gamma D^T x^{(i)} + 2\left(f - \sum_k u_k^{(i+1)} \right) \right)$

Optimization Solution using ADMM - $x^{(i+1)}, \rho^{(i+1)}$

$$\mathcal{L}(v,x,\rho) = \beta \sum_{j=1}^{N} \phi(|x_j|;b) - \langle \rho, x - Dv \rangle + \frac{\gamma}{2}\|x - Dv\|_2^2 + \left\| f - \left(v + \sum_k u_k \right) \right\|_2^2$$

- $x^{(i+1)} = \operatorname{argmin}_{x\in\mathbb{R}} \left\{ \beta \sum_{j=1}^{N} \phi(|x_j|;b) - \sum_{j=1}^{N} \rho_j x_j + \frac{\gamma}{2} \sum_{j=1}^{N} \left(x_j - (Dv)_j\right)^2 \right\}$
- $x_j^{(i+1)} = \operatorname{argmin}_{x_j\in\mathbb{R}} \left\{ \beta\phi(|x_j|;b) - \rho_j x_j + \frac{\gamma}{2} \left(x_j - (Dv)_j\right)^2 \right\}$, for $j = 1, \dots, N$
- $x_j^{(i+1)} = \operatorname{argmin}_{x_j\in\mathbb{R}} \left\{ \mu\phi(|x_j|;b) + \frac{1}{2} \left(x_j - (Dv)_j - \frac{\rho_j}{\gamma}\right)^2 \right\}$, for $j = 1, \dots, N$,

$$\phi''(t;b) = \begin{cases} -b, t \in [0, \sqrt{2/b}\,) \\ 0 \quad , t \in [\sqrt{2/b}\, + \infty) \end{cases}$$

where $\mu = \frac{\beta}{\gamma}$.

$-b\mu + 1$, for $|x_j| < \sqrt{2/b}$

1 , for $|x_j| \geq \sqrt{2/b}$

They are strongly convex if and only if sufficient conditions [4]:

$$b < \frac{1}{\mu} \Longrightarrow \gamma > b\beta \Longrightarrow \gamma = \tau b\beta, \text{for } \tau \in \mathbb{R}, \tau > 1$$

$$x_j^{(i+1)} = \min\left(\max\left(\frac{1}{1-\mu b} - \frac{\frac{\mu\sqrt{2b}}{1-b\mu}}{|h|}, 0 \right), 1 \right) h_j \text{, where } h_j = \left(Dv^{(i+1)}\right)_j + \frac{\rho_j^{(i)}}{\gamma}$$

- $\rho^{(i+1)} = \rho^{(i)} - \gamma\left(x^{(i+1)} - Dv^{(i+1)}\right)$

[4] M. Huska, A. Lanza, S. Morigi and I. Selesnick, "A convex-nonconvex variational method for the additive decomposition of functions on surfaces," in Inverse Problems, vol. 35, no. 12, pp. 124008, 2019.

Input parameters for JMD

$$\mathcal{L}(\{u_k\},\{\omega_k\},v,x,\rho) = \alpha\sum_{k=1}^{K}\left\|\partial_t\left[u_k^{+}(t)e^{-j\omega_k t}\right]\right\|_2^2 + \beta\int_0^\infty \phi(|x(t)|;b)dt - \langle\rho(t),x(t)-\partial_t v(t)\rangle + \frac{\gamma}{2}\|x(t)-\partial_t v(t)\|_2^2 + \left\|f(t)-\left(\sum_{k=1}^{K}u_k(t)+v(t)\right)\right\|_2^2$$

K: The total number of AM-FM modes to be decomposed.

- Large K: lead to duplicate modes.
- Small K: mode mixing problems.

α: The bandwidth of the decomposed AM-FM modes.

- High α: incorrect noisy modes (narrow bandwidth).
- Low α: mode mixing issues (wide bandwidth).

β: A regularization parameter affecting the extracted jump component.

- High β: fail to reconstruct the full jump component.
- Low β: include unwanted oscillatory artifacts.
- $\beta \approx \frac{1}{\text{(number of expected jump)}}$.

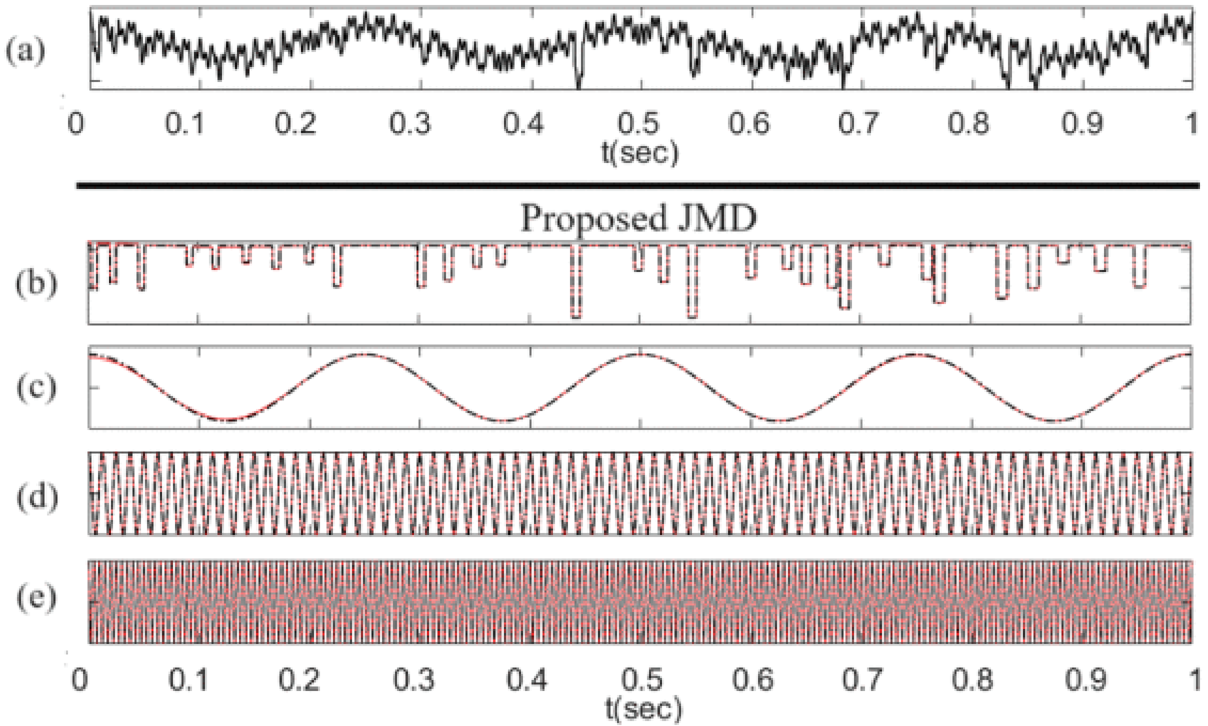
$\bar{b}$: The expected minimal jump height. ($b = 2/\bar{b}^2$)

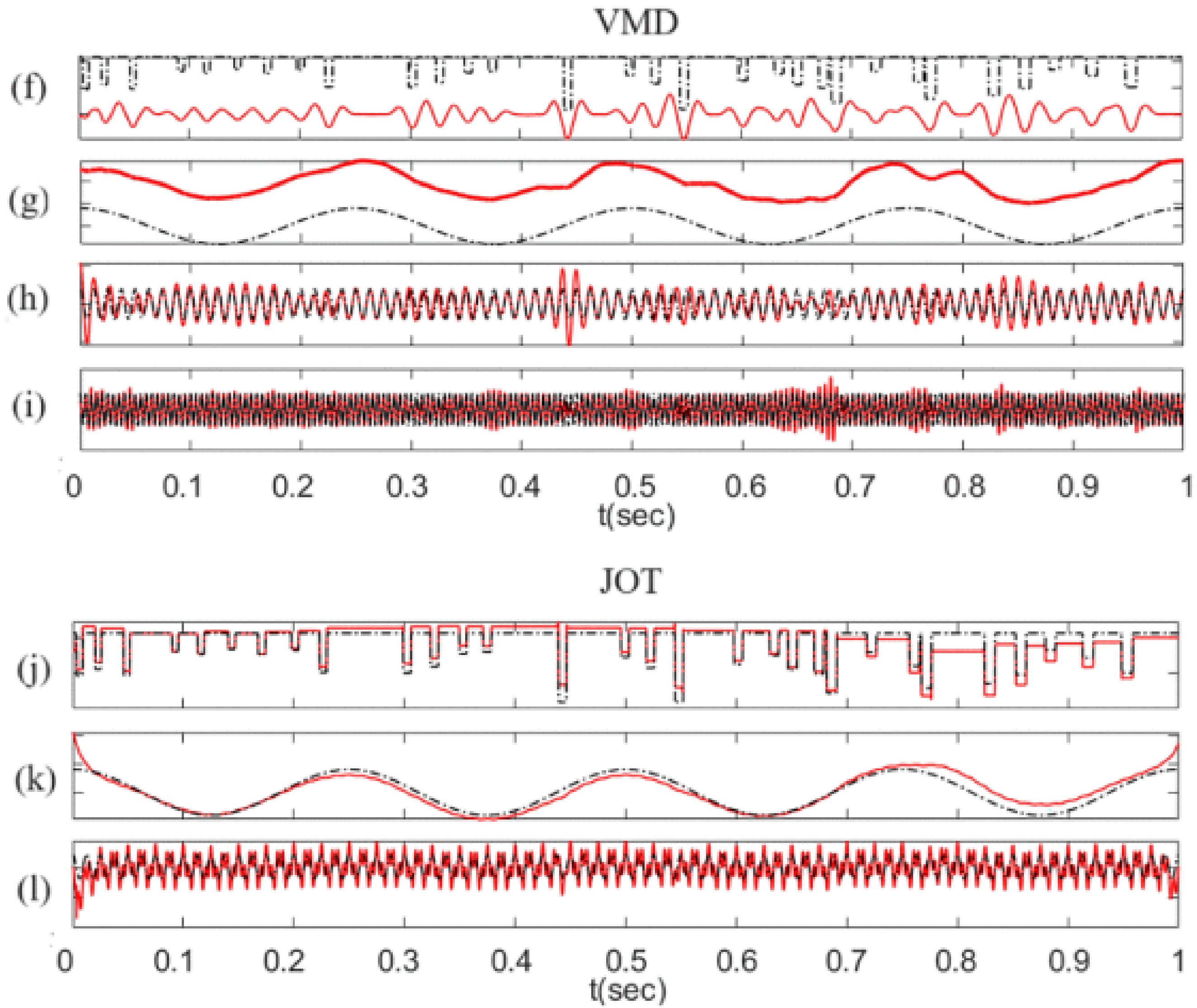
- Jumps with $\left|(Dv)_j\right| > \bar{b}$ are equally penalized.

τ: A scalar parameter used for initializing γ.

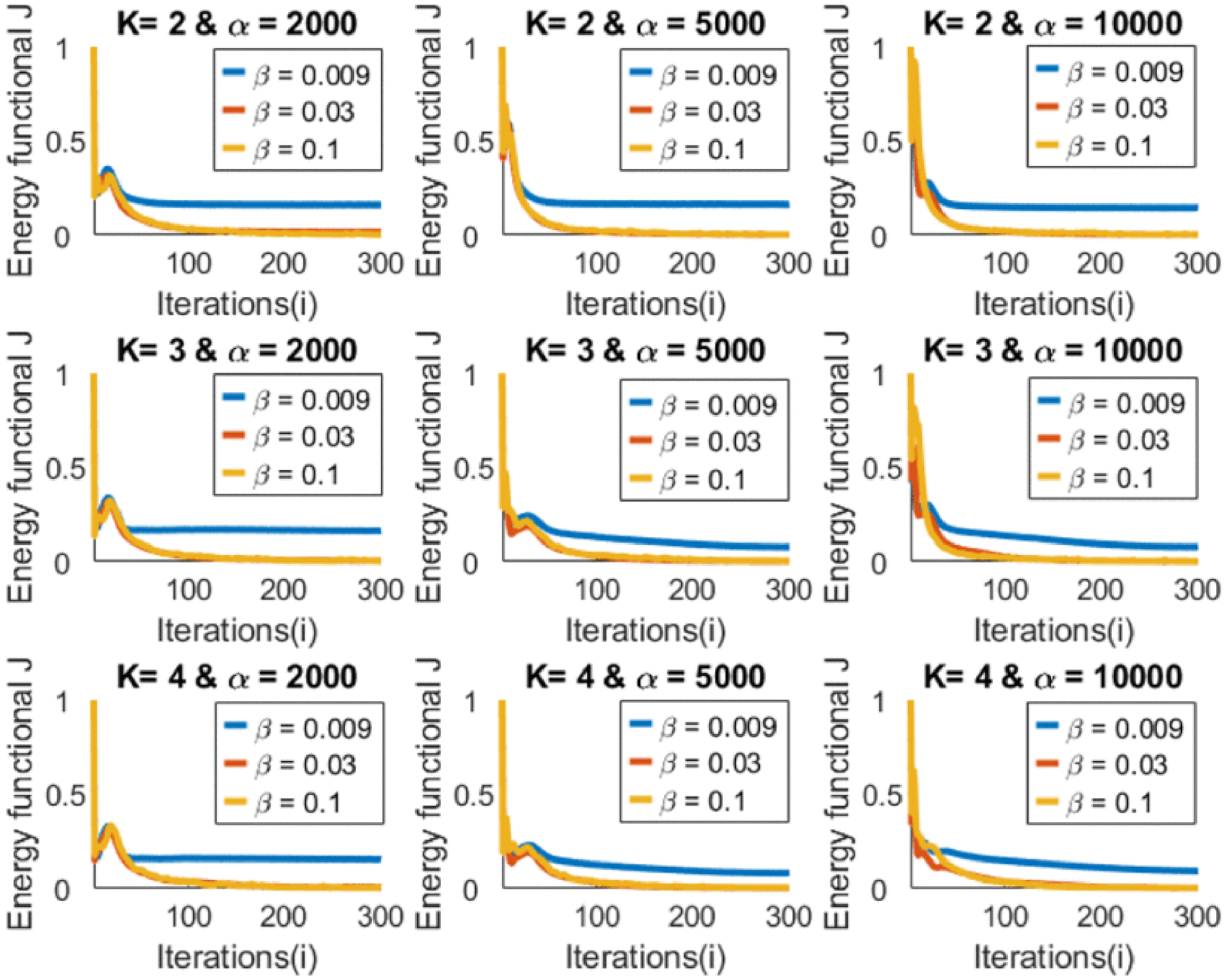
Experiments
Synthetic Signal

$$f(t) = \cos(2\pi 4t) + \cos(2\pi 80t) + \cos(2\pi 200t) + v(t) + \eta$$

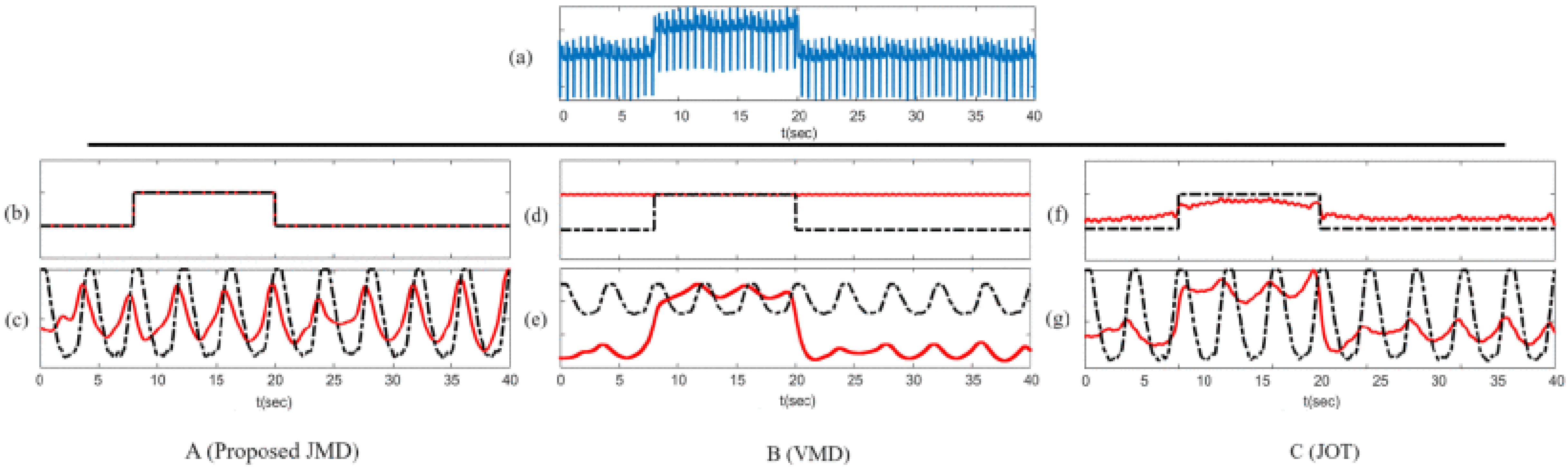




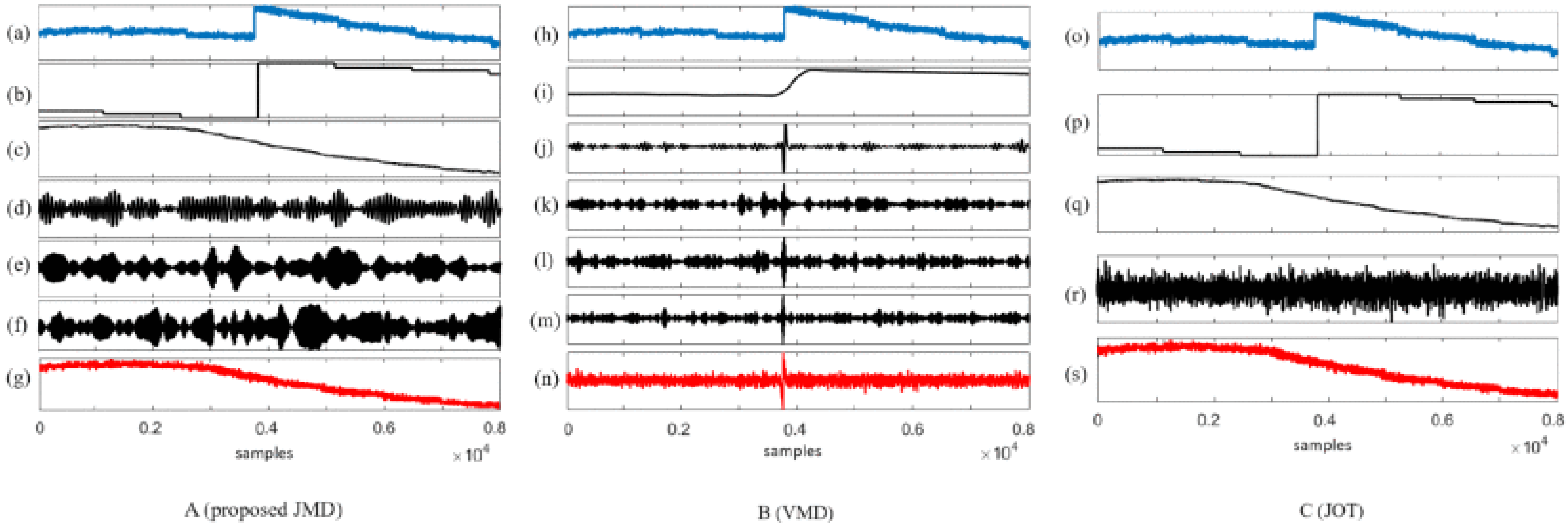
Empirical evidence of numerical convergence



Experiments
Electrocardiogram (ECG)-Derived Respiration (EDR)



Experiments - Earth's Magnetic and Electric Field (EMF)



Multivariate Extension

- The model is given as follows:

$$\mathbf{f}(t) = \sum_{k=1}^{K} \mathbf{u}_k(t) + \mathbf{v}(t) + \mathbf{n}(t)$$

- $\mathbf{f}(t)$: A multivariate signal consisting of C number of data channels $\mathbf{f}(t) = [f_1(t), \dots, f_C(t)]$.
- $\mathbf{u}_k(t)$: The set of AM-FM decomposed modes.
- $\mathbf{v}(t)$: The multivariate jump component.
- $\mathbf{n}(t)$: The multivariate noise component.

Multivariate Extension

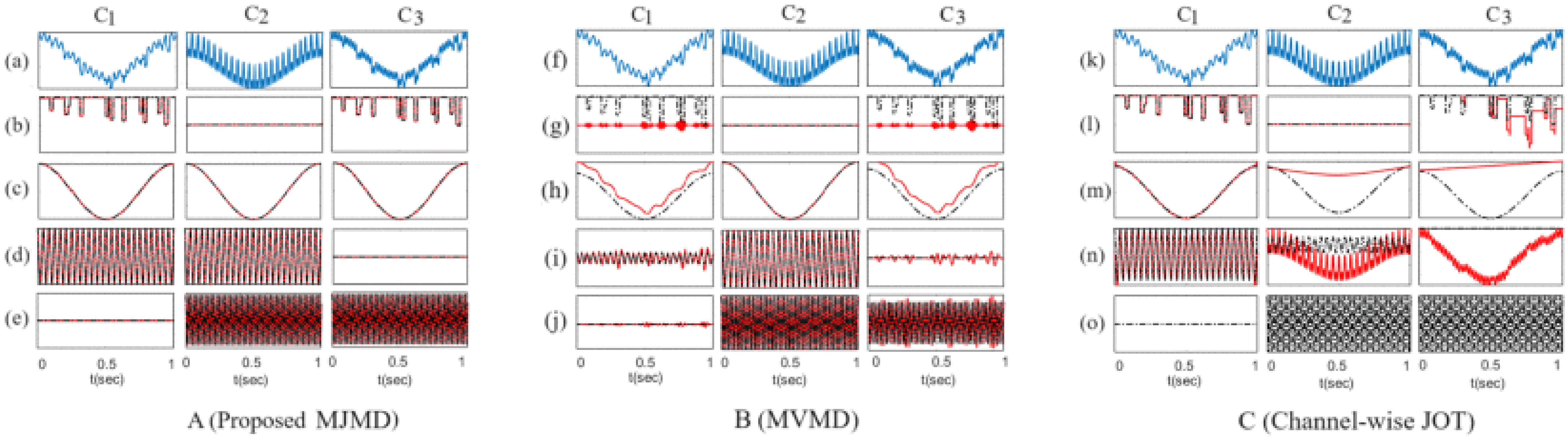
- The goal of the multivariate extension of the JMD method is to extract the jump components $\mathbf{v}(\mathrm{t})$ in all channels and the AM-FM modes $\mathbf{u}_k(\mathrm{t})$ from input signal $\mathbf{f}(\mathrm{t})$.

$$J_1 = \sum_k \sum_c \left\| \partial_t \left[u_{k,c}^{+}(t) e^{-j\omega_k t} \right] \right\|_2^2$$

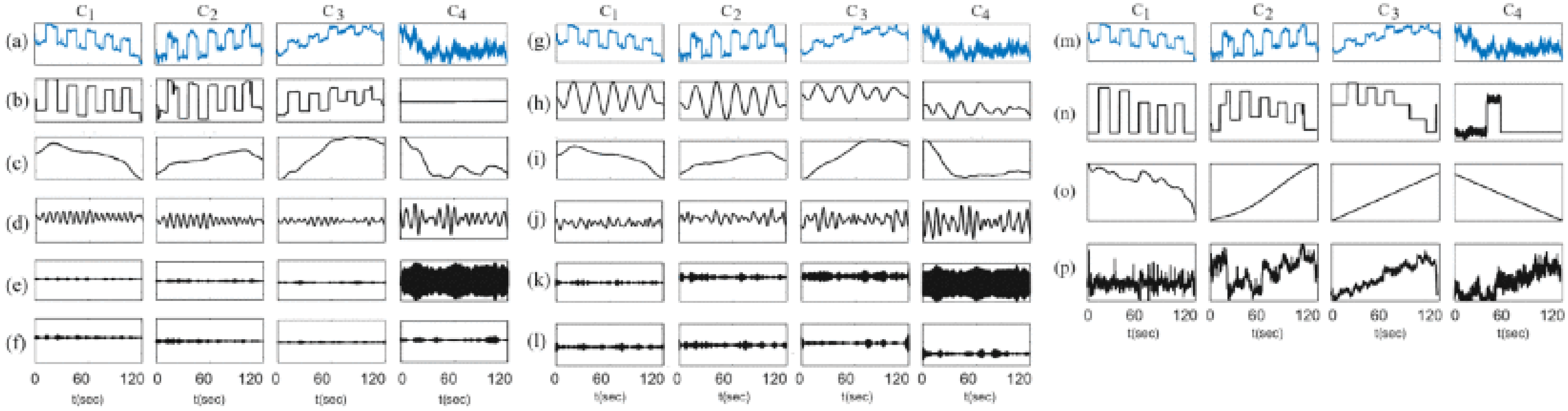
$$J_2 = \int_0^\infty \phi(|\partial_t v_c(t)|; b)dt$$

- The AM-FM modes will be aligned in frequency in the multivariate algorithm.

Experiments - synthetic multivariate signal



Experiments - Electroencephalogram (EEG) signal



A (Proposed MJMD) B (MVMD) C (Channel-wise JOT)

Conclusions

- JMD is based on a variational optimization formulation solved using the alternating direction method of multipliers (ADMM).
- A decomposition method, JMD, deal with the signal as composed of wide-band jumps and narrow-band AM-FM signals.
- A multivariate algorithm, multivariate JMD (MJMD), is also proposed.
- Real-life data, including EEG, ECG, and Earth’s electric field signals are demonstrated the broad applicability of these approaches across different fields.

References

[1] A. Cicone, M. Huska, S. -H. Kang and S. Morigi, "JOT: A Variational Signal Decomposition Into Jump, Oscillation and Trend," in IEEE Transactions on Signal Processing, vol. 70, pp. 772-784, 2022.

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