Chromatic Derivatives and Approximations

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Outline

- Motivation
- Chromatic Derivatives
- Chromatic Approximations
- Result
- Conclusion

Nyquist theorem

The expansion of a signal

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) sinc(t-n)$$

is *global* in nature, because it requires samples of the signal at integers of arbitrarily large absolute value.

Taylor series

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}$$

The Taylor expansion is of *local* nature.

Such a series converges uniformly on every finite interval, and its truncations provide good local signal approximations.

Motivation

However, unlike the Shannon expansion, the Taylor expansion has found very limited use in signal processing.

Disadvantage of Taylor series:

- its truncations have rapid error accumulation.
- Numerical evaluation of higher order derivatives of a function given by its samples is very noise sensitive.
- Compare to Shannon expansion, lose linear shift invariant operator.

$$A[f](t) = \sum_{n=-\infty}^{\infty} f(n)A[sinc](t-n)$$

Chromatic Derivatives

Consider normalizing and scaling the Legendre polynomials

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n\,(w)P_m\,(w)dw = \delta(n-m)$$

Define operator polynomials

$$K^n(w) = (-j)^n P_n^L\left(j\frac{d}{dt}\right)$$

We call operators K^n the chromatic derivatives associated with the Legendre polynomials.

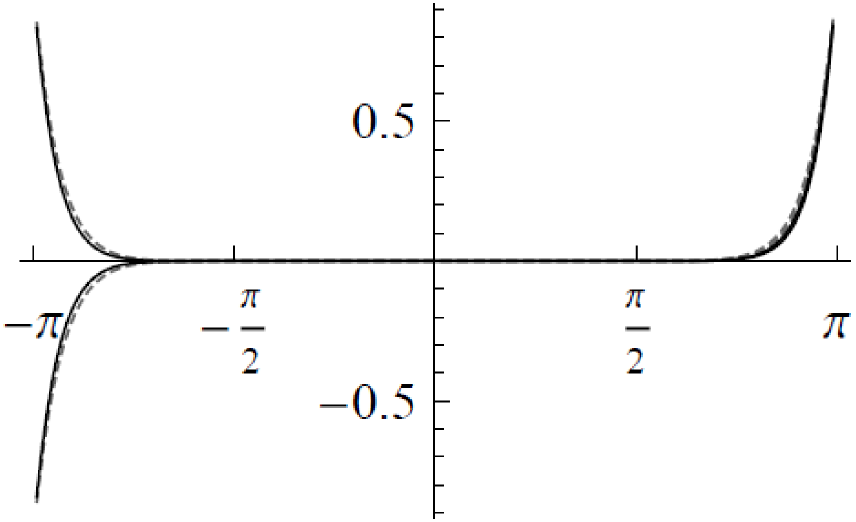
$$K^n[e^{iwt}] = (j)^n P_n^L(w)e^{iwt}$$

Numerical differentiation of band limited signals

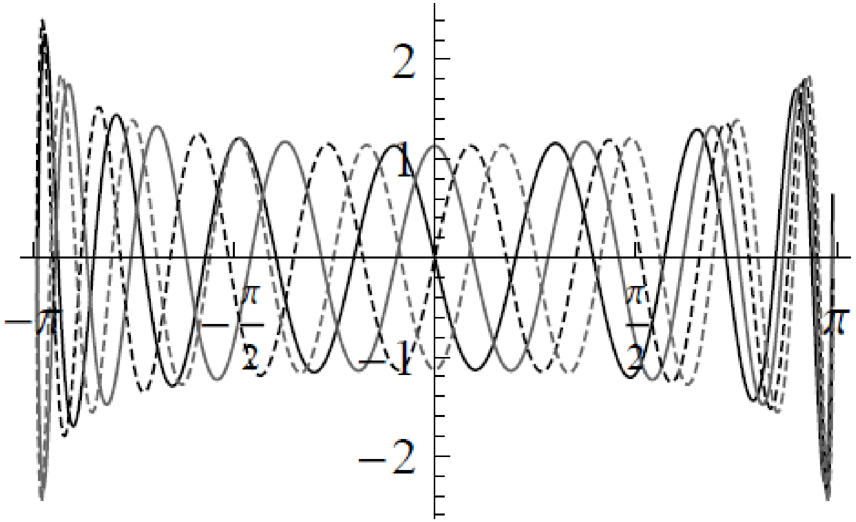
differentiation in Fourier domain

n=15 to n=18

$$f^{(n)}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (jw)^n \hat{f}(w) e^{jwt} dw \qquad K^n[f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (j)^n P_n^L(w) \hat{f}(w) e^{iwt} dw$$



(normalized) derivatives



chromatic derivatives

Chromatic Expansion

Chromatic expansion of associated with the Legendre polynomials

$$f(t) = \sum_{n=0}^{\infty} K^n[f](u)K^n[sinc(t-u)] = \sum_{n=0}^{\infty} K^n[f](u)\sqrt{2n+1}j_n(\pi(t-u))$$

Where $j_n(.)$ is the spherical Bessel function of the first kind of order *n*.

Chromatic expansion

$$f(t) = \sum_{n=0}^{\infty} K^n[f](0)\sqrt{2n+1}j_n(\pi t)$$

The coefficients of the Nyquist expansion of a signal

$$K^n[f](0) = \sum_{n=-\infty}^{\infty} f(t)\sqrt{2n+1}j_n(\pi t)$$

Chromatic Expansions and Approximations in General

Consider families of orthonormal polynomials $p_n(w)dw$ with weight function w(ω)

$$d\mu = \mathrm{w}(w)dw$$

$K^n(w) = (-j)^n p_n\left(j\frac{d}{dt}\right)$, $K^n[e^{iwt}] = (j)^n p_n(w)e^{iwt}$

We define that

$$B_0(t) = \int_{\mathbb{R}} e^{jwt} d\mu = \int e^{jwt} \mathrm{w}(w)dw$$

$$f(t) = \sum_{n=0}^{\infty} (-1)^k K^n[f](u) K^n[B_0(t-u)]$$

$$app[f,n,u](t) = \sum_{n=0}^{\infty} (-1)^k K^n[f](u) K^n[B_0(t-u)]$$

Chromatic Expansion Examples

Chebyshev polynomials of the first kind

$$B_0(t) = \int e^{jwt} \frac{2}{\sqrt{\pi^2 - w^2}} dw = J_0(\pi t)$$

Where $J_0(.)$ the Bessel function of the first kind and of order 0.

Since

$$K^n[J_0(\pi(t))] = (-1)^n \sqrt{2} J_n(\pi t)$$

The corresponding chromatic expansion is the Neumann series

$$f(t) = f(u)\, J_0(\pi(t-u)) + \sqrt{2} \sum_{n=1}^{\infty} K^n[f](u) K^n[J_n(\pi(t-u))]$$

Chromatic Expansion Examples

Hermite polynomials

$$B_0(t) = \int e^{jwt} e^{-w^2} dw = e^{-t^2/4}$$

Since

$$K^n\left[e^{-t^2/4}\right] = \frac{(-1)^n t^n}{\sqrt{2^n n!}} e^{-t^2/4}$$

The corresponding chromatic expansion is back to Taylor expansion of $f(t)e^{t^2/4}$, multiplied by $e^{-t^2/4}$.

Recurrence Relation

Since families of orthonormal polynomials satisfy a recurrence of the form

$$p_{n+1}(w) = \frac{w}{\gamma_n} p_n(w) - \frac{\gamma_{n+1}}{\gamma_n}\ p_{n+1}(w)$$

corresponding differential operators K^n satisfy the recurrence

$$K^{n+1} = \frac{1}{\gamma_n}\left(\frac{d}{dt} \circ K^n\right) + \frac{\gamma_{n-1}}{\gamma_n} K^{n-1}$$

where the recursion coefficients for a given Weight

- Change of basis of chromatic derivatives

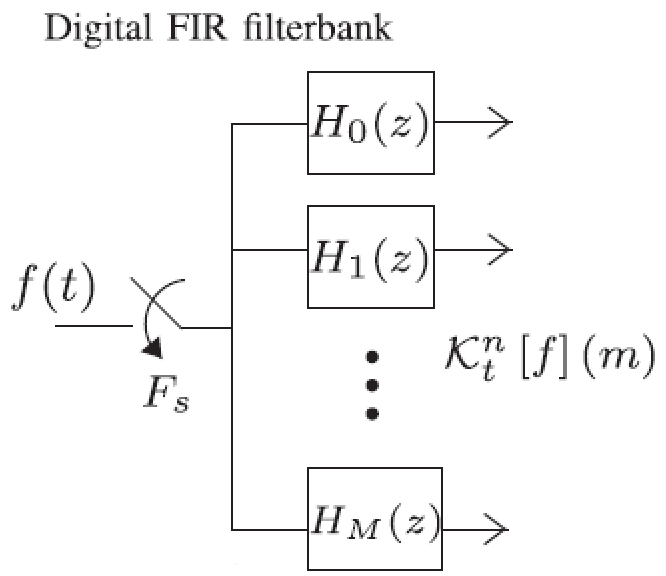
$$K_{p_2}^{n-1} = \sum C\,(\gamma) K_{p_1}^{n-1}$$

- Computation

Filter Bank

Since orthogonality of provide perfect reconstruction

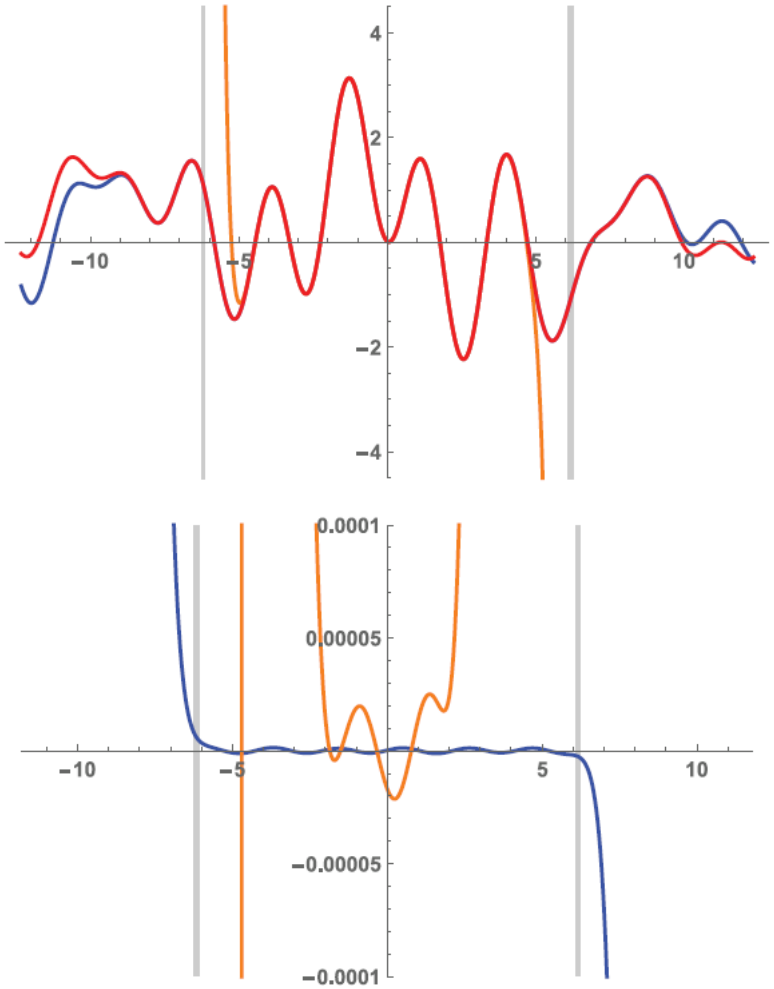
$$H_k\left(e^{j\omega}\right) = (j)^k P_k^L(w)$$



Approximation Result

Top: Taylor’s approximation of order 31 (orange) of a band limited signal (red) and its chromatic approximation of the same order (blue);

Bottom: the corresponding errors of the Taylor approximation (orange) and of chromatic approximation (blue).



Conclusion

chromatic approximation provide

- Robust to noise
- Bounded
- Good local performance
- linear shift invariant operator

Reference

1. A. Ignjatovic, C. Wijenayake, and G. Keller, “Chromatic derivatives and approximations in practice - part i: A general framework,” *IEEE Trans. on Signal Processing*, vol. 66, no. 6,pp. 1498–1512, 2018.
2. A. Ignjatovic, C. Wijenayake, and G. Keller, “Chromatic derivatives and approximations in practice - part ii: Nonuniform sampling, zero-crossings reconstruction and denoising,” *IEEE Trans. on Signal Processing*, vol. 66, no. 6, pp. 1513–1525, 2018.
3. A. Ignjatovic, “Chromatic derivatives and local approximations,” *IEEE Trans. on Signal Processing*, vol. 57, pp. 2998–3007, 2009
4. A. Ignjatovic, “Chromatic derivatives, chromatic expansions and associated spaces,” *East Journal of Approximations*, vol. 15, pp. 263–302, 2009.