**Tutorial for Chromatic Derivatives**

**彩色微分的介紹**

**Author: Jen-Chieh Cheng**

**Edit by Jian-Jiun Ding**

鄭任傑 著

丁建均老師編輯

2020.04

1. CONTENTS

[CONTENTS 2](#_Toc102213843)

[Chapter 1 INTRODUCTION 5](#_Toc102213844)

[1.1 Chromatic Derivatives 5](#_Toc102213845)

[1.2 Chromatic Expansion 7](#_Toc102213846)

[1.3 Recurrence relationship 12](#_Toc102213847)

[1.4 Examples 13](#_Toc102213848)

[1.5 Chromatic Derivative for Filter Banks 16](#_Toc102213849)

[Chapter 2 MORE GENERAL INTEGRAL TRANSFORMS 20](#_Toc102213850)

[2.1 General Concepts 20](#_Toc102213851)

[2.1.1The chromatic expansion 20](#_Toc102213852)

[2.2 Examples 21](#_Toc102213853)

[Chapter 3 MULTIDIMENSIONAL CHROMATIC DERIVATIVES 31](#_Toc102213854)

[3.1 Orthogonal Polynomials in Two Variables 31](#_Toc102213855)

[3.1.1Product Orthogonal Polynomials 32](#_Toc102213856)

[3.1.2Orthogonal Polynomials on the Unit Disk 32](#_Toc102213857)

[3.2 Orthogonal Polynomials in Several Variables 33](#_Toc102213858)

[3.3 Multidimensional Chromatic Derivatives 36](#_Toc102213859)

[3.3.1The Bargmann–Segal–Foch Space 38](#_Toc102213860)

[3.3.2The Bargmann transform 39](#_Toc102213861)

[3.3.3Chromatic Expansions in the Bargmann–Segal–Foch Space 39](#_Toc102213862)

[Chapter 4 Chromatic derivatives and expansions with weights 41](#_Toc102213863)

[4.1.1Chromatic Expansion 42](#_Toc102213864)

[4.2 Poisson wavelet transform 42](#_Toc102213865)

[4.3 Examples with Varying weights 46](#_Toc102213866)

[4.3.1Laplace transform, Laguerre-type weighs 46](#_Toc102213867)

[4.3.2Bargmann transform, Hermite-type weights 48](#_Toc102213868)

[Chapter 5 Future Work 50](#_Toc102213869)

[Reference 51](#_Toc102213870)

[Appendix 52](#_Toc102213871)

# INTRODUCTION

Chromatic derivatives and the chromatic series expansions of bandlimited signal were introduced by Ignjatovic in [4]-[7] and later generalized and applied the various integral transforms in [12]-[15]. In the signal processing, the filter banks of chromatic derivatives were proposed in [6], [9]-[10] for bandlimit signal, and in [11] for the non-bandlimit signal. Furthermore, in Chapter 3, multidimensional chromatic derivatives were proposed in [8], [14], and [17]. In the Chapter 4, chromatic derivatives and expansions with varying weights were provided in [17].

In the Chapter 1, we briefly review the chromatic derivatives and the chromatic expansions which were based on the Fourier transform with the Legendre, Chebyshev, Hermite polynomials. The advantages of the chromatic expansions provide robustness to noise, good local performance and linear shift invariant operator which are important property for signal processing. And in practical signal processing application, the filter banks of chromatic derivatives were also introduced. Next, in the Chapter 2, we generalized the Fourier transform to the Laplace, Hankel, Fourier sine transform, Fourier cosine transform, with the others orthogonal polynomials, e.g., the Gegenbauer polynomial, the Laguerre polynomials. Moreover, in the Chapter 3, we review multidimensional the chromatic derivatives and the chromatic expansions. Finally, in the Chapter 4, we review the extension of the chromatic derivatives and the chromatic expansions with varying weights.

## Chromatic Derivatives

In signal processing, there are several signal expansion methods, including the Nyquist theorem:



is *global* in nature.

and the Taylor series:



is *local* in nature.  
The chromatic derivative is to apply polynomial expansion in the frequency domain.

Let  be the Legendre polynomial of order *n*. Consider normalizing and scaling the Legendre polynomials [19]:

, where ,

then

,

we can define the operator polynomials



We call the operator *the chromatic derivatives associated with the Legendre polynomials*.

The chromatic derivatives on  is:



Note that  is an eigenfunction of *Kn*. In , we apply the fact that



for most of the orthogonal polynomials.

Compared to derivatives and chromatic derivatives in the Fourier domain

**• Derivatives**

.   
Here, we use  to denote the Fourier-Stieltjes transform of *f*(*t*).

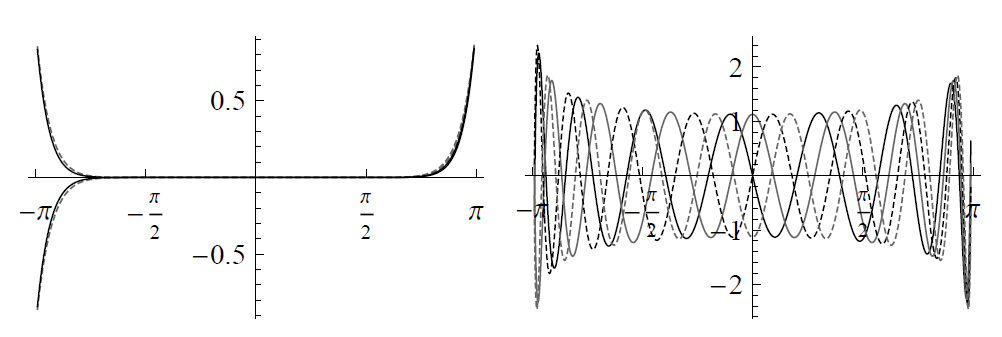
 or    
Eq. is the special case where  for |*ω*| < *π* and  otherwise.

**• Chromatic derivatives**

,



Interestingly, from a frequency domain perspective, the chromatic derivatives can be interpreted as an expansion of the orthogonal polynomials.



(a) (b)

Fig. 1 (a) (normalized) derivatives; (b) chromatic derivatives (acts as a polynomial in the frequency domain). (Q1: Add the citation of the figure)

1. A. Ignjatovic, “Chromatic derivatives and local approximations,” *IEEE Transactions on Signal Processing*, vol. 57, pp. 2998–3007, 2009.

## Chromatic Expansion

Consider families of orthonormal polynomials *P*𝑛(*ω*)𝑑*ω* with weight function *w*(*ω*)

 (i.e., ).

We define that

,

.

Then we have

**[Formula of Chromatic Expansions]:**



This is the chromatic approximation of *f*(*t*) of order *n*, centered at *u*. The way to compute *Kn*[*f*](*u*) can be seen from (1-6). Specially, if *w*(*ω*) is a rectangular function (i.e., the weight function of the Legendre polynomial):

,   
then

.

Compared to the Taylor expansion in (1-2),  is replaced by  and  is replaced by .

**(Proof):** (A) For the case where *u* = 0, we can apply the Fourier-Stieltjes transform:

.

We have

, . Since the transfer functions *Pn*(*ω*) of the chromatic derivatives in the Fourier domain are orthonormal polynomials, we consider the Fourier-Stieltjes transform of a signal can be expanded in

 where .

Therefore,





where *w*(*ω*) is the weight function of the Legendre polynomial in (11). From (15) and (17),

,   
then



where *B*0(*t*) is defined in .

(B) For the case where *u* ≠ 0, consider expanding the following function:

,   
 where .

,

,   
Then, from Eq. ,

.



**[Property 1.1] Parseval’s Property:**



(Proof): From , we have



. #

Specially, if the chromatic derivative is associated with the (rescaled and normalized) Legendre polynomials (i.e., *w*(*ω*) is defined in ), then



**[Property 1.2] Convolution Property:**



(Proof): From (23), we have

   
 . #

Specially, if the chromatic derivatives are associated with the Legendre polynomials (i.e., *w*(*ω*) is defined in ), then is reduced to



## Recurrence relationship

Orthonormal polynomials satisfy a recurrence of the form as follows:



for some positive coefficients *γn.*

We define , . Let us extend our definition of *Kn* by letting *Kn* [*f*(*t*)] = 0 for all integers *n* < 0 and satisfy the recurrence.



## Examples

We then show the examples of Chromatic Expansions using some orthogonal polynomials. Some examples are extracted from [4] which have more details.

**[Example 1] (Legendre polynomials):**

If we consider , as in , then



,

where .

(.) is Bessel function of the first kind, and *j*(.) is the spherical Bessel function of the first kind. The last equality holds since, from Eq. (18.17.19) and Eq. (10.47.3) in [1],

.

Then becomes



**[Example 2] (Chebyshev polynomials of the first kind):**

If we consider

, for *n* ≠ 0, .   
where *Tn* is the *n*th order Chebyshev polynomial of the 1st kind:

,

   
where *c*0 = 1 and , then







The last equality is from Eq. (10.9.2) in [1]

.

Then becomes

.

**[Example 3] (Hermite polynomials):**

If we consider

,

where *Hn*(*x*) is the Hermite polynomial:

,   
then





.   
The last equality applies . Then, from the fact that [20]

,

, ,   
where  means the inverse Fourier transform, we have

,

.   
Therefore, from ,

.

Examples 1, 2, and 3 are the chromatic derivatives using the Legendre polynomials, the Chebyshev polynomials of the 1st order, and the Hermite polynomial, respectively.

## Chromatic Derivative for Filter Banks



Fig. 2

1. M. Narasimha, A. Ignjatovic, and P. Vaidyanathan, “Chromatic derivative filter banks,” *IEEE Signal Processing Lett*., vol. 9, no. 7, pp.215–216, Jul. 2002.

Suppose that *x*(*t*) and its Fourier transform *X*(*ω*) are expanded by

, .

Also suppose that, in Fig. 2, the analysis and synthesis filters satisfy the biorthogonality condition:



(Proof of the perfect reconstruction (PR) property):

Note that

   
where \* means the convolution and *hm*(*t*) is the inverse Fourier transform of *Hm*(*ω*). Therefore,

, . Consider the output of the *m*th analysis filter evaluated at *t* = 0,



.

In and , there are many possible choices for *Hk*(*ω*) and *Fk*(*ω*). Specially, when applying Taylor series expansion:

   
then in and ,

, , ,

   
Then, from, the fact that [20]

   
we can verify that *Fm*(*ω*) and *Hk*(*ω*) in and satisfy .

In fact, one can apply the chromatic derivative to design the filter bank. For example, one can apply the Legendre polynomial in Example 1 and the Chebyshev polynomial of the 1st type in Example 2.

**[Legendre Filter Bank]**

One can also apply the Legendre polynomial to design the filter band. From ,





where  means the rectangular function

.   
Then, it is no hard to show that the equality in holds.

**[Chebyshev Filter Bank]**

Similarly, from and *Hk*(*ω*) and *Fk*(*ω*) in can be defined as:





**For Non-bandlimited Signals**

For the non-bandlimited case, instead of , the reconstruction constraint in the frequency domain can be redefined as

.

Note that, different from (46), the support of integral is (−∞, ∞). In this case, we can apply (39) to define the Hermite filter bank.

**[Hermite Filter Bank]**





# MORE GENERAL INTEGRAL TRANSFORMS

## **General Concepts**

In Chapter 1, we introduced the chromatic derivatives with various orthonormal polynomials, but only consider the case based on the Fourier transform. There are still other integral transforms worth considering. Therefore, in Chapter 2, we generalize chromatic derivatives to apply more general integral transforms and combine many different transformations and some different orthogonal polynomials to get a more generalized result.

Assume that  and  satisfy the following two conditions:

1. 
2.  be such that 
3. If , we further assume that , for all  converge uniformly.

Consider the integral transform

, 

or

,

### The chromatic expansion

The chromatic expansion of the *f*



where .

Proof:

, , ,

.

Therefore, it can be expanded in

#.

Moreover, the kernel function can be expanded in

.

## Examples

Some examples are extracted from [12] which have more details.

**[Example 1] The Laplace transform using the Gegenbauer polynomial:**

The one-sided Laplace transform is defined as

,  , .

In this example we take .

Consider the Gegenbauer polynomial (also called the ultraspherical polynomial) defined by

,

where , , .

The Gegenbauer polynomials are orthogonal with respect to the weight function  , and orthogonality

.

Note that the Legendre polynomial is a special case of the Gegenbauer polynomial where *v* = 1/2 and the Chebyshev polynomial of the 1st kind is a special case of the Gegenbauer polynomial where *v* = 0.

By changingin the integrand,

,

where .

We set ,

,

where .

By formula 9 in [2, p. 171],

, where *I* is the modified Bessel function of order *v* defined by

 .

The chromatic expansion of



And become to

, , , which is consistent to

.

**[Example 2] The Laplace transform using the Laguerre polynomial:**

As same as Example 1, consider Laplace transform , , . But take . Laguerre polynomials , , The Laguerre polynomials are orthogonal with respect to the weight function  , and orthogonality



In this case, consider the normalized Laguerre polynomials 

By [2, p. 175, formula 31]

,

where .

The chromatic expansion of

.

And becomes

,

, , which is consistent to the generating function of the Laguerre polynomials

.

**[Example 3] The Hankel Transform with the Laguerre polynomial**

The Hankel transform of order *v* is defined as

.

(Q3: Please re-check it. It is different from the standard definition of the Hankel transform.)

Note that some common definition of the Hankel transform , or, where,. Back to ,  is a solution of the differential equation . But in this case, we consider derivative of , 

, since asymptotic forms for the Bessel functions 

Set , , of ,



In this case, consider the normalized Laguerre polynomials 

, 

.

Chromatic derivative is denoted by .

By formula 2 in [3, p. 42],

,

The chromatic expansion of

.

And become to



.

**[Example 4] The Hankel transform of order 0 using the Legendre polynomial:**

Consider the Hankel transform of order 0:

, , .

Consider the Legendre polynomials which satisfy the orthogonality relation

,

set , shift interval to.

,

, .

Chromatic derivative is denoted by .

By formula1 in [3, p. 13],

,

The chromatic expansion of

.

And become to

.

**[Example 5] The Fourier sine transform using the shifted Legendre polynomial:**

The Fourier sine transform is defined as

,  , .

Since , we may change  to , and let .

We adopt the shifted Legendre polynomial:

,

and shift interval to .

Consider the following sine transform



which is the Fourier sine transform of,

,

.

The chromatic expansion of



Where

,

since [2, p. 94, Formula 2], and become



**[Example 6] The Fourier cosine transform using the Gegenbauer polynomial:**

The Fourier cosine transform is defined as

,  , , .

In this example we take ,



Consider the Gegenbauer polynomial defined by



For , reduce to



Then, we have



We apply to ,

.

We denote normalized polynomials, then the CD

,

where  , then apply the transform.

The chromatic expansion of



, where



And becomes



which is consistent to

.

# MULTIDIMENSIONAL CHROMATIC DERIVATIVES

In previous chapters, many variations of the one-dimensional CD were introduced. In this chapter, we generalize the one-dimensional CD to the multi-dimensional CD. We first review the two-dimensional orthogonal polynomials, and to the multi-dimensional orthogonal polynomials. Next, combine it with the CD to produce new results. More details were shown in [8], [18].

## Orthogonal Polynomials in Two Variables

First, we introduce some space of the two-dimensional orthogonal polynomials briefly.

**[Example 1]**

Space of polynomials of total degree n is denoted by

, 



**[Example 2]**

The homogeneous polynomials of degree *n* are denoted by

, .



Let {*pk*} and {*qk*} be sequences of orthogonal polynomials with two weight functions *w*1 and *w*2, respectively. Then a mutually orthogonal basis with respect to *W* is given by





are orthonormal.

### Product Orthogonal Polynomials

We introduce some 2D product orthogonal polynomials in the Cartesian coordinate.

**Product Hermite polynomials**

, , .

**Product Laguerre polynomials**

, , .

**Product Jacobi polynomials**

, , 

### Orthogonal Polynomials on the Unit Disk

We review some 2D product orthogonal polynomials in the polar coordinate. Consider the unit disk .

, 



The Gegenbauer polynomial is defined as

, , .

.

,

where , , , and is the Pochhammer symbol, also called the shifted factorial, defined as following

, ,

according to different orthonormal basis generate various forms.

## Orthogonal Polynomials in Several Variables

Let *N* denotes the set of nonnegative integers, and be a multi-index ,

We define the notation

, , and 

Let , we define the monomial , and  is the degree of .

The set of all polynomials in *d* variables and the degree at most *n* is denoted by

, .

**[Example 1]**:

.

The set of homogeneous polynomials of degree *n* is denoted by

, .

**[Example 2]**:



Every polynomial in d variables can be written as a linear combination of homogenous polynomials

.

Since natural order of several variables does not exist, we use the lexicographic order as following  if the first non-zero entry in the difference  is positive, e.g.  of degree 3 is ordered in front of  of degree 6. A polynomial *P* is called an orthogonal polynomial if *P* is orthogonal to all polynomials of lower degree; i.e.,  for all  with deg *Q* < deg *P*. Denote by  the space of orthogonal polynomials of degree exactly n:



For each multisequence we define a linear functional on  by .

Let the elements of the set  be arranged as  according to the lexicographic order. Let **x**n denote the column vector



whose elements are arranged in the lexicographic order.

Define the vector moments ,  which is a matrix of size and has elements  for |*α*| = *k*, |*β*| = *j*. If  is a sequence of polynomials in we get the column polynomial vector  , where is the lexicographic order in .

**[Definition]** Let  be a moment functional. A sequence of polynomials  in is said to be orthogonal with respect to  if

 for *n* *>* |*m*| and,

where *sn* is an invertible matrix of size .

By definition  if and only if  for |*α*| = *n*, |*β*| = *m*. Hence, each is orthogonal to any polynomial of lower degree. It is known that if  is a moment functional and  is orthogonal as defined before, then  is a basis for . Hence, there exists matrices **c***k* of size such that

.

And .

We adopt the following notation, If  is a polynomial in ,then the polynomial  will be denoted by , where *xα* is replaced by  where *α* is a multi-index. More explicitly, if ,  , , is replaced by

.

Let  denote the sequence of orthonormal polynomials with respect to . Let  denote the space of all square integrable functions with respect to the weight function *W*. For any function , consider its generalized Fourier expansion with respect to ,



, where .

If we use the vector notation, we have

, where ,

where **c***n* is a column vector with components  and |*α*| = *n*.

## Multidimensional Chromatic Derivatives

**[Definition]** The *n*th chromatic derivatives:

.

Let , and .

The weighted inverse Fourier transform is

.

Since

,

We have

,

**[Property]** The Parseval theorem of *n*th chromatic derivatives:

,

,

,

use the vector notation .  are analytic on , and satisfy .

**[Chromatic Series]**:

The multidimensional chromatic expansion of a signal *f* is denoted as

.

**[Example** **1]**

If the support of  is the half space,we consider the normalized Laguerre polynomials

, , .

### The Bargmann–Segal–Foch Space

**[Definition] The Bargmann–Segal–Foch Space**

,

where .

The inner product can also be defined in terms of the Taylor series coefficients of *F* and *G*. For, if ,  are multi-indices.

 ,,

where , and .

Then, we have

,

where  can be simplify to

.

Hence, the inner product of *F*, *G* can be rewritten as

.

The set is an orthonormal basis in the Bargmann–Segal–Foch Space,

.

### The Bargmann transform

**[Definition]**

The Bargmann transform of a function  is defined by

.

We define the n-dimensional normalized Hermite polynomial of degree *m* by, where , , with weight function. The *n*-dimensional normalized Hermite functions are defined by .The Bargmann transform of the normalized Hermite function  is .

**[Property]** Since the Hermite functions  are an orthonormal basis of  , we have

.

### Chromatic Expansions in the Bargmann–Segal–Foch Space

We denote 

,

Then, we have .

**[Definition]**

We define the *α*th chromatic derivative of *F*(*z*) with respect to the operator *L* and the Hermite polynomials as

.

**[Chromatic Series]**

There exists a function *φ* whose Bergmann transform has the property that its chromatic derivatives  are an orthogonal basis of . Hence, any  can be written in the form

.

.

# Chromatic derivatives and expansions with weights

Let  be a kernel function and an integral transform on some function spaces, with a measure *ν* on a finite or infinite interval (*c*, *d*), a function



Original assumption , or if , . The motivation for the modification is to get expansions around more points. Substituting the assumption by a weaker one, let  be any point for which  has finite many sign changes, we can extend the notion of chromatic derivatives and expansions to a wider family of kernel functions



Let *L* be a linear differential operator:



Let  be an arbitrary polynomial. According to



Let  such that  on (*c*, *d*), let us denote by



The *n*th orthonormal polynomial  with respect to 



The *n*th chromatic derivative of *f* with respect to *x*0 and *w* at *x* is



If we write  ,

.

Thus has the formal Fourier series

.

Let  ,

.

### Chromatic Expansion

Therefore, *f* has the formal chromatic expansion

.

## Poisson wavelet transform

More details were shown in [17]. Poisson distribution is defined by

.

The Poisson wavelet is defined by

, ,

and the admissibility constant associated with  is

.

The kernel of the wavelet transform is denoted

.

The wavelet transform of a function is defined by

.

The wavelet transform of the function :

.

(Proof): More detail can be shown in [17].

We take derivative of *un* with respect to *b* , and *a*





We denote that



Since

, and .

then



Hence, we have.

Let  be a fixed point.  has one sign change at , where .

We divide the domain of the integration to two parts, If *a*0 > 0 then  and , and

 ,.

For *i* = 1, 2 letbe positive weights with , such that has finite moments

.



The *m*th chromatic derivative of  .

Let . Let , and

, where .

Thus, the expansion of *f*

,

where , .

Then, we denote that

,

.

Hence .

By Parseval’s formula



We get



That is,  has an orthogonal expansion with respect to 



And



Since  is in  as a function of *y*,



## Examples with Varying weights

Some examples are extracted from [17] which have more details.

### Laplace transform, Laguerre-type weighs

, , , ,

, ,

where are the orthonormal Laguerre polynomials. If , then , and it degenerate to the Example 2 in the Chapter 2.2. We denote  in this example. Thus, the of 

,

We have  , 

,



where  are denote the Laplace transform.

Therefore, we have the formal chromatic expansion of 



s are no longer orthogonal.

### Bargmann transform, Hermite-type weights

The Bargmann transform was introduced in the Chapter 3.3.2 and is defined as

.

The corresponding differential operator is

.

Let *α* be a multiindex, *α* = *k*1, *k*2, …, *kn*, and  is a polynomial with *n* variables. It is easy to see that

.

Let a weight function, such that are positive and have finite moments. Let  , and



Let  be the orthonormal polynomials with respect to, and  .

Thus, we have the orthogonality



The *α*th chromatic derivative of *F* with respect to at *z* is defined as

,

and

.

Let us denote



and



By using the , , and , then we have the chromatic expansion of *F*



# Future Work

In this paper, we review the chromatic derivatives have been proposed with combination of many different integral transforms and orthogonal polynomials. Some varying weight and the filter bank of the chromatic derivatives also been provided. We hope to extent the chromatic derivatives to more general integral transforms and orthogonal polynomials and physic meaning corresponding to the general from. In signal processing, the advantages of the chromatic expansions were good at noise resistant and local performance. However, there are room for improvement on the issue of the chromatic derivatives, e.g., more efficient practical method. In the future, we hope to apply more general transforms or orthogonal polynomials to the chromatic derivatives which have more efficient method.

1. Reference
2. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.28 of 2020-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
3. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, vol. I, Bateman Manuscript, New York: Mc-Graw-Hill, 1955.
4. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, vol. II, Bateman Manuscript, New York: Mc-Graw-Hill, 1954.
5. A. Ignjatovic, “Chromatic derivatives and local approximations,” *IEEE Transactions on Signal Processing*, vol. 57, pp. 2998–3007, 2009.
6. A. Ignjatovic, “Chromatic derivatives, chromatic expansions and associated spaces,” *East J. Approximations*, vol. 15, no. 3, pp. 263–302, 2009. (The detail deriving in [5] is in this paper).
7. A. Ignjatovic, C. Wijenayake, and G. Keller, “Chromatic derivatives and approximations in practice - part i: A general framework,” *IEEE Trans. on Signal Processing*, vol. 66, no. 6, pp. 1498–1512, 2018.
8. A. Ignjatovic, C. Wijenayake, and G. Keller, “Chromatic derivatives and approximations in practice - part II: Nonuniform sampling, zero-crossings reconstruction and denoising,” *IEEE Trans. on Signal Processing*, vol. 66, no. 6, pp. 1513–1525, 2018.
9. A. Ignjatovic and A. I. Zayed, “Multidimensional chromatic derivatives and series expansions,” in *Proceedings of the American Mathematical Society*, vol. 139, no. 10, pp. 3513–3513, 2011.
10. M. Narasimha, A. Ignjatovic, and P. Vaidyanathan, “Chromatic derivative filter banks,” *IEEE Signal Processing Lett*., vol. 9, no. 7, pp.215–216, Jul. 2002.
11. P. Vaidyanathan, A. Ignjatovic, and M. Narasimha, “New sampling expansions of bandlimited signals based on chromatic derivatives,” in *Proc. 35th Asilomar Conf. Signals, Syst. Comput*., Monterey, CA, pp. 558–562, 2001.
12. G. Walter and X. Shen, “A sampling expansion for non-bandlimited signals in chromatic derivatives,” *IEEE Trans. Signal Process*., vol. 53, no. 4, pp. 1291–1298, Apr. 2005.
13. A. I. Zayed, “Generalizations of chromatic derivatives and series expansions,” *IEEE Trans. Signal Process*., vol. 58, no. 3, pp. 1638–1647, Mar. 2010.
14. A. Zayed, “Chromatic derivatives of generalized functions,” *J. Integral Transforms Spec. Funct*., vol. 22, no. 4, pp. 383–390, 2011.
15. A. Zayed, “Chromatic expansions and the Bargmann transform,” in *Multiscale Signal Analysis and Modeling*, X. Shen and A. Zayed, Eds. New York, NY, USA: Springer, 2013.
16. A. Zayed, “Chromatic derivatives of generalized functions,” *J. Integral Transforms Spec*. *Funct*., vol. 22, no. 4, pp. 383–390, 2011.
17. A. I. Zayed, “Chromatic expansions in function spaces,” *Transactions of the American Mathematical Society*, vol. 366, no. 8, pp. 4097–4125, 2014.
18. A. P. Horvath. “Chromatic derivatives and expansions with weights”, 2016. [Online]. Available: http://arxiv.org/abs/1601.06135.
19. C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*. Cambridge: Cambridge University Press, 2014.
20. D. G. Zill, W. S. Wright, and J. J. Ding, *Engineering Mathematics*, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019.
21. R. N. Bracewell, *The Fourier Transform and Its Applications*, 3rd ed., McGraw Hill, Boston, 2000.
22. Appendix

In general, the orthogonality of orthogonal polynomials  with respect to a weight is as following

.

The table of some classical orthogonal polynomials is as following

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Chebyshev, | | Legendre, | Hermite, |
| Formula |  | |  |  |
| Domain of orthogonality |  | |  |  |
| Weight |  | |  |  |
| Square of norm, |  | |  |  |
|  | Laguerre, | Gegenbauer, | | |
| Formula |  |  | | |
| Domain of orthogonality |  |  | | |
| Weight |  |  | | |
| Square of norm, |  |  | | |

目前 Chapter 1 已改成易看得懂的方式，並已檢查確定每個 equations 的正確性

(Q4): Chapter 2 以後都改成可以看得懂的敘述方式，以讓數學還不錯的專題生能讀懂為目標

(Q5): 仔細檢查每個 equations 是否正確

完成

(Q6): 內文加上 citation numbers

完成

(Q7): 其實 chromatic derivatives = orthogonal polynomial expansion in the frequency domain，照這個觀念來整理，就會較清楚且不致於太繁雜

(Q8): 除了 filter bank 之外，是否有其他的應用？可以簡述

(Q9): Chapter 5 的定位不清楚，且內容不完整，可以再補齊

編入第一章

(Q10): Equations 要標號，其中 Chapter 2 以後要重新編號 完成

(Q11): 格式：排版不要太擠

Equations 靠中間

要用 [Theorem \*.\*], [Property \*.\*] 來標明要闡述的概念

完成

(Q12): 要用較完整的句子來敘述

完成

(Q13): 了解正確的文法用法並使用之

http://disp.ee.ntu.edu.tw/class/%E8%AB%96%E6%96%87%E8%8B%B1%E6%96%87%E5%B8%B8%E8%A6%8B%E7%9A%84%E5%95%8F%E9%A1%8C.ppt

(Q14): 加上一個 Appendix, Summary of Orthogonal Polynomials，對每一種內文中有提到的 orthogonal polynomial 列出

(i) Formula, (ii) weight function, (iii) support, (iv) orthogonality property

完成