Fast and Accurate Spherical Harmonics Products

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Outline

- Spherical Harmonics Product
- New Method
- Conclusion

The Laplace's Equation

$$\nabla^2 f = 0$$

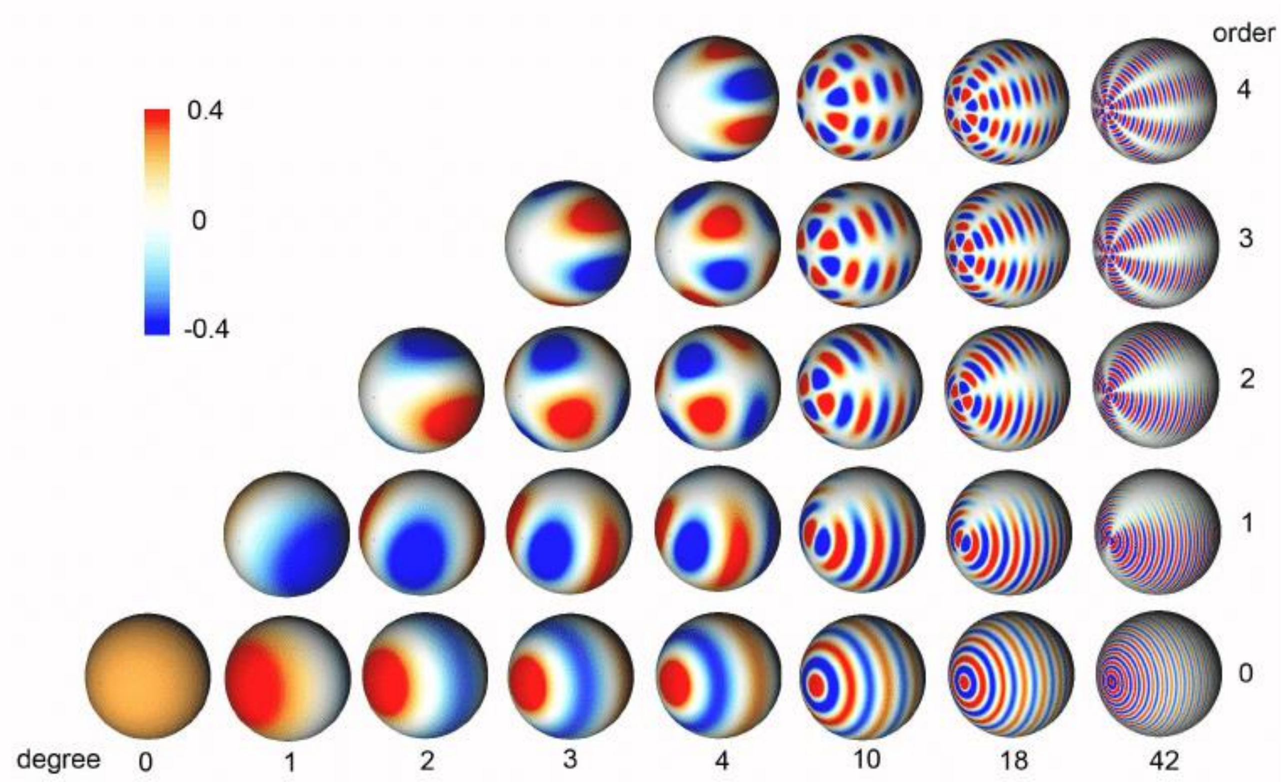
$$\nabla^2 f \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}f\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}f$$

The angular part solution of the PDE is the spherical harmonics

$$Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta)e^{im\phi}$$

Band n, mode m. for n ≥ 0 and −n ≤ m ≤ n.

Spherical Harmonics



M. Chung, K. Dalton, and R. Davidson, “Tensor-based cortical surface morphometry via weighted spherical harmonic representation,” *IEEE Trans. Med. Imag.*, vol. 27, no. 8, pp. 1143–1151, Aug. 2008.

Spherical Harmonics

- $Y_n^m(\theta,\phi)$ Orthonormal Basis on the sphere.
- Spherical Harmonics Reconstruction

$$F(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{n,m}\, Y_n^m(\theta,\phi) = \sum_{i=0}^{\infty} f_i\, B_i$$

- Spherical Harmonics Projection

$$f_{n,m} = \int f(\theta,\phi)[Y_n^m(\theta,\phi)]^*\, \mathrm{d}\Omega = \int F B_i$$

Rendering

- 3-Dimensional Field (light audio).
- Relighting Precomputed Radiance Transfer, Shadow Fields.

$$L_0(x, \omega_o) = \int L(x, \omega_i)\, \rho(\omega_i, \omega_o) d\omega$$

Spherical Harmonics Product

- Spherical Harmonics Double Product

$$\int F_1(\omega)F_2(\omega)d\omega = \int \sum_i f_{1,i}B_i(\omega) \sum_i f_{2,i}B_i\,(\omega)d\omega = \sum_i f_{1,i}f_{2,i} = \boldsymbol{f}_1 \cdot \boldsymbol{f}_2$$

- Spherical Harmonics Triple Product

$$\int \underbrace{F_1(\omega)F_2(\omega)}_{G(\omega)} F_3(\omega)d\omega = \sum_i \underbrace{\sum_j \sum_k C_{i,j,k} f_{1,i}f_{2,j}}_{g_k = \boldsymbol{f}_1 \otimes \boldsymbol{f}_2} f_{3,k} = \boldsymbol{f}_1 \otimes \boldsymbol{f}_2 \cdot \boldsymbol{f}_3$$

where $C_{i,j,k} = \int B_i(\omega)B_j(\omega)B_k(\omega)\, d\omega$

- Complexity $O(n^6)$

Spherical Harmonics Product

- Spherical Harmonics Multiple Product

$$\int \underbrace{F_1(\omega)F_2(\omega)\cdots F_{k-1}(\omega)}_{G(\omega)} F_k(\omega)d\omega$$

$$= \underbrace{\sum\sum\cdots\sum C_{j_1,j_2\cdots j_k} f_{1,i}f_{2,j}\, f_{3,k}}_{g_k=\otimes_k(f_1,f_2,\cdots,f_{k-1})}$$

$$= \otimes_k\, (\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_{k-1}) \cdot \boldsymbol{f}_k$$

where $C_{j_1,j_2\cdots j_k} = \int B_{j_1}(\omega)B_{j_2}(\omega)\cdots B_{j_k}(\omega)\, d\omega$

- Complexity $O\left(n^{2k}\right)$

Spherical Harmonics and 2D Fourier Series

- 2D Fourier Series

$$F(\theta, \phi) = \sum_{-n<s,t<n} f^*_{s,t} e^{i(s\theta + t\phi)}$$

- Rewrite SH to Fourier Series

$$Y_l^m(\theta, \phi) = \sum_{s,t} y^*_{s,t} e^{i(s\theta + t\phi)}$$

- Conversion from SH to Fourier series

$$F(\theta, \phi) = \sum_{l=0}^{n-1} \sum_{m=-n}^{n} f_{l,m} Y_l^m(\theta, \phi) = \sum_{s,t} f^*_{s,t} e^{i(s\theta + t\phi)}$$

- Conversion Complexity $O(n^3)$

Fourier Series Product

- Spherical Harmonics Triple Product

$$G(\theta,\phi) = F_1(\theta,\phi)F_2(\theta,\phi) = \left(\sum_{s_1,t_1} f^*_{1,s_1,t_1} e^{i(s_1\theta+t_1\phi)}\right)\left(\sum_{s_2,t_2} f^*_{2,s_2,t_2} e^{i(s_2\theta+t_2\phi)}\right)$$

$$G(\theta,\phi) = \sum_{s,t} g^*_{s,t} e^{i(s\theta+t\phi)}$$

$$g^*_{s,t} = \sum_{\substack{s_1+s_2=s \\ t_1+t_2=t}} f^*_{1,s_1,t_1} f^*_{2,s_2,t_2} \Leftrightarrow \boldsymbol{g}^* = \boldsymbol{f}^*_1 * \boldsymbol{f}^*_2$$

where $*$ denotes the 2D convolution operator.

- Could be solved by FFT

Fast SH Multiple Product

$$\int \underbrace{F_1(\omega)F_2(\omega)\cdots F_{k-1}(\omega)}_{G(\omega)} F_k(\omega)d\omega = \underbrace{\otimes_k\left(\boldsymbol{f}_1, \boldsymbol{f}_2, \cdots, \boldsymbol{f}_{k-1}\right)}_{g} \cdot \boldsymbol{f}_k$$

$$G(\theta,\phi) = \sum_{s,t} g^*_{s,t} e^{i(s\theta + t\phi)}$$

$$\boldsymbol{g}^* = \boldsymbol{f}^*_1 * \boldsymbol{f}^*_2 * \cdots * \boldsymbol{f}^*_{k-1}$$

- Could be solved by FFT

Fast SH Multiple Product

$$\boldsymbol{g}^* = \boldsymbol{f}_1^* * \boldsymbol{f}_2^* * \cdots * \boldsymbol{f}_{k-1}^*$$

- Straight-forward way

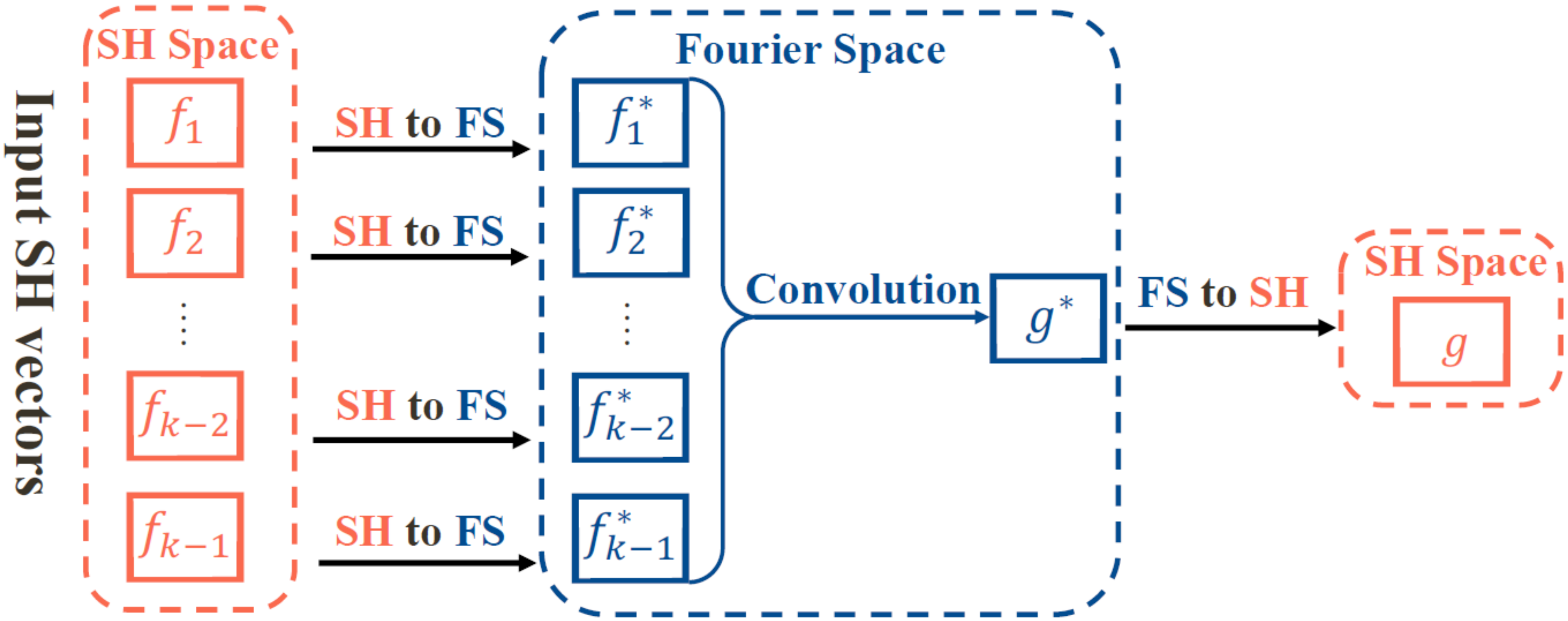
$$\boldsymbol{g}^* = \underbrace{\underbrace{\left[\underbrace{(\boldsymbol{f}_1^* * \boldsymbol{f}_2^*)}_{deg.\,2n-1}\right] * \boldsymbol{f}_3^*}_{deg.\,3n-2} * \cdots * \boldsymbol{f}_{k-1}^*}_{deg.(k-1)(n-1)+1}$$

- Divide-and-conquer

$$\boldsymbol{g}^* = \underbrace{\underbrace{\underbrace{(\boldsymbol{f}_1^* * \boldsymbol{f}_2^*)}_{deg.\,2n-1} \cdots \underbrace{(\boldsymbol{f}_{\lfloor \boldsymbol{k}/2 \rfloor - 1}^* * \boldsymbol{f}_{\lfloor \boldsymbol{k}/2 \rfloor}^*)}_{deg.\,2n-1}}_{deg.\left\lfloor \frac{\boldsymbol{k}}{2} \right\rfloor (n-1)+1} * \underbrace{\underbrace{(\boldsymbol{f}_{\lfloor \boldsymbol{k}/2 \rfloor + 1}^* * \boldsymbol{f}_{\lfloor \boldsymbol{k}/2 \rfloor + 2}^*)}_{deg.\,2n-1} \cdots \boldsymbol{f}_{\boldsymbol{k}-\mathbf{1}}^*}_{deg.\left\lceil \frac{\boldsymbol{k}}{2} \right\rceil (n-1)+1}}_{deg.(k-1)(n-1)+1}$$

- Complexity $O\left(k^2 n^2 log(kn)\right)$

Fast Spherical Harmonics Product



H. Xin, Z. Zhou, D. An, L.-Q. Yan, K. Xu, S.-M. Hu, and S.-T. Yau, “Fast and accurate spherical harmonics products,” *ACM Trans. on Graph.*, vol. 40, no. 6, pp. 1–14, 2021.

Complexity

- Traditional: $O\left(n^{2k}\right)$
- New Method: $O\left(kn^3 + k^2n^2log(kn)\right)$

SH to FS: $O(kn^3)$ + FS convolution: $O\left(k^2n^2log(kn)\right)$

Conclusion

New Approach provide

- Simplifying from Spherical Harmonics Product to Convolution.
- Fast Spherical Harmonics Product through FFT in the Fourier Space.

Reference

1. H. Xin, Z. Zhou, D. An, L.-Q. Yan, K. Xu, S.-M. Hu, and S.-T. Yau, “Fast and accurate spherical harmonics products,” *ACM Trans. on Graph.*, vol. 40, no. 6, pp. 1–14, 2021.
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