Interesting Medium's Articles 1

Riemann's Rearrangement Theorem

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} ... = \mathit{ln}(2)$$

However, if $(1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + ... = \mathit{ln}(2)$

$$\Rightarrow \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + ... = \mathit{ln}(2)$$

$\star$ Riemann's rearrangement theorem : manipulating the order in a given infinite series, we can end up with any desirable results.

$\Rightarrow$ infinite series hold true only when considered for a given particular order and for any other order their prediction stops making sense.

$\Rightarrow$ the commutative property doesn't work for infinite series.

Reference

From the Simpsons to Riemann's paradox: discussing Riemann's rearrangement theorem.
https://medium.com/geekculture/from-the-simpsons-to-riemanns-paradox-discussing-riemann-s-rearrangement-theorem-ed2a3bfe707f

Feynman's Integration Trick

Procedure :

1. 想辦法將積分內的單變數函數變成有用的雙變數函數
2. 想辦法得到兩邊都爲 t 的函數的等式
3. 等式兩邊對 t 做微分、不定積分或是定積分
4. 可以得到結果或甚至得到比原問題更推廣化的結果

How to calculate the factorial integral : $\int_0^\infty x^n e^{-x} dx$?

1. 從我們已知的下手: $\int_0^\infty e^{-x} dx = 1$
2. 想辨法將積分内的函數變成雙變數函數: $\int_0^\infty te^{-tx} dx = 1$
3. 將等式兩邊都變成 t 的函數: $\int_0^\infty e^{-tx} dx = \frac{1}{t}$
4. 兩邊對 t 微分: $\int_0^\infty \frac{\partial}{\partial t} e^{-tx} dx = \frac{d}{dt}\frac{1}{t} \Rightarrow \int_0^\infty -xe^{-tx} dx = -\frac{1}{t^2}$

   $\rightarrow$ Generalization : $\int_0^\infty x^n e^{-tx} dx = \frac{n!}{t^{n+1}}$

   $\rightarrow$ 將 t 代 1 即可得到 $\int_0^\infty x^n e^{-x} dx = n!$

How to calculate the Dirichlet integral : $\int_0^\infty \frac{sinx}{x} dx$?

1. 想辦法將積分內的函數變成雙變數函數: 考慮 $I(t) = \int_0^\infty \frac{sinx}{x} e^{-tx} dx$
   $\Rightarrow$ 好處: 積分內函數對 t 微分可以將分母的 x 消掉
2. 對 t 微分: $I'(t) = \int_0^\infty -sinxe^{-tx} dx$
3. 實際用分部積分算出該積分結果: $I'(t) = \int_0^\infty -sinxe^{-tx} dx = -\frac{1}{t^2+1}$
   $\rightarrow$ 可以得到兩邊都是 t 的函數的等式
4. 兩邊對 t 作不定積分: $I(t) = \int_0^\infty \frac{sinx}{x} e^{-tx} dx = -arctan(t) + C$
5. 求 C : t 代 $\infty \rightarrow 0 = -\frac{\pi}{2} + C \rightarrow C = \frac{\pi}{2}$
6. $\int_0^\infty \frac{sinx}{x} e^{-tx} dx = \frac{\pi}{2} - arctan(t)$ $\rightarrow$ t 代 0，可得 $\int_0^\infty \frac{sinx}{x} dx = \frac{\pi}{2}$

How to calculate the Gaussian integral : $I = \int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx$?

1. 想辦法將積分內的函數變成雙變數函數: 考慮 $G(t) = \int_0^{\infty} \frac{e^{-t^2(1+x^2)}}{1+x^2} dx$
   $\Rightarrow$ 好處: 積分內函數對 t 微分後可以得到與 I 相關的結果
2. 對 t 微分: $G'(t) = -2te^{-t^2} \int_0^{\infty} e^{-(tx)^2} dx = -2e^{-t^2} \int_0^{\infty} e^{-u^2} du$
   $$\Rightarrow G'(t) = -Ie^{-t^2}$$ $\rightarrow$ 得到兩邊都是 t 的函數的等式
3. 兩邊對 t 做定積分: $\int_0^{\infty} G'(t)dt = \int_0^{\infty} -Ie^{-t^2} dt \Rightarrow \int_0^{\infty} G'(t)dt = -\frac{I^2}{2}$
   $$\Rightarrow \lim_{b\to\infty} G(b) - G(0) = -\int_0^{\infty} \frac{1}{1+x^2} dx = -\frac{I^2}{2}$$
   $$\Rightarrow -\lim_{x\to\infty} arctan(x) = -\frac{\pi}{2} = -\frac{I^2}{2}$$
   $$\Rightarrow I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Reference

Mastering The Amazing Feynman Trick
https://www.cantorsparadise.com/mastering-the-amazing-feynman-trick-d896c9a494e6

The Implicit Information Hypothesis

Information = (bits, gates)

- Data is communicated between a transmitter and a receiver with the bit being its basic unit
- Gates(plus a clock and some wiring) compose a machine that processes or transforms the bits
- For any particular piece of information there may exist multiple configurations of bits and gates that perform the same computation and therefore represent the same information

- Axiom : The fundamental laws of physics of our universe can in theory be approximated by digital simulation to such a degree that it produces the same phenomena as we are familiar with in our universe
- Explicit information : a non-redundant description of the states of all elementary particles, plus a non-redundant description of the machine that repeatedly applies the laws of physics to these states.

  $\rightarrow$ Explicit information is a complete and non-redundant description of our universe.

- Explicit circuit : it mimics the fundamental laws of physics and runs simulation of our universe
- Explicit matrix : it represents all the bits of the explicit information of the simulated universe, i.e. the complete state of the simulated universe

However, we cannot see macroscopic phenomena like stars or planets by looking only at the changing bits of explicit information because from the explicit information point of view, the information of macroscopic phenomena is not necessary in the non-redundant description

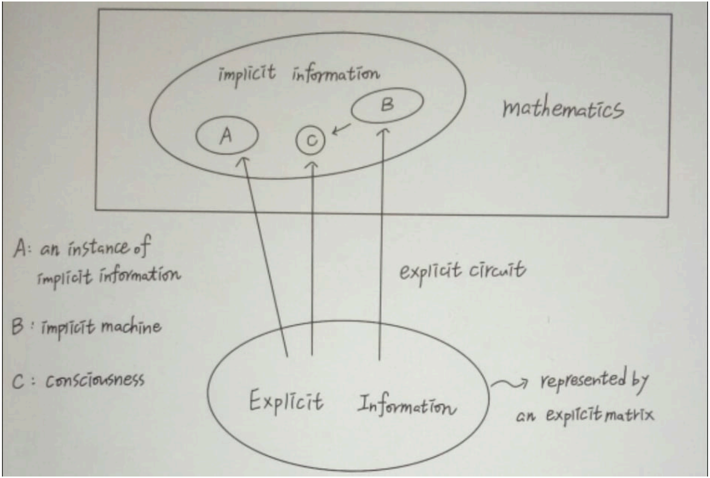
- Implicit information : the set of all instances of implicit information
- Instance of implicit information : a sequence of transformations of explicit information
- Transformation : a machine which accepts explicit information (or an instance of implicit information) as input, and produces an instance of implicit information as output.
- Implicit machine : an instance of implicit information that reads the bits of the explicit matrix, performs some transformation of these bits, and outputs its result to its own implicit matrix.
    - Living organisms are implicit machines that replicate, feed and survive in an entropic environment.
    - Views of galaxies, stars, planets and clouds are also examples of implicit matrices.

Stories told by implicit machines

- the fluctuating stock market price of a company
- the collective behavior of cells in a multi-cellular organism
- a burning flame behaves as if it wants to survive, wandering around in search for fuel
- the process of evolution : a breadth-first search for survival machines, with the tree of life being the ongoing result of that search
- ......

- Mathematics : the set of all possible information, i.e., the set of all configurations of bits and gates. It is a fixed set that never changes. Implicit information is the subset of mathematics. When the bits of explicit information change, so does its implicit information, which instantly becomes the newly selected subset of mathematics
- Implicit Information Hypothesis : Implicit information is as true and real as the explicit information it derives from. Our reality is that perspective and space, the mathematical space where all transformations of explicit information actually exists.

- Implicit Consciousness Hypothesis : Consciousness is an instance of implicit information that is written or read by an implicit machine. Being conscious is the same as being the implicit machine.
    - Given just the explicit information of our universe, it would take transformations to bring into view what a brain is thinking, therefore the information of conscious thought exists only in the mathematical space
    - Joscha Bach : We exist inside the story that the brain tells itself.


implicit information
B
A
C
mathematics
A: an instance of implicit information
explicit circuit
B : implicit machine
C : consciousness
Explicit Information
represented by an explicit matrix

- This hypothesis is not a theory of everything. It does not explain the origin of explicit information, and its philosophical claims and interpretations are not testable, so it cannot be called a theory
- Mathematics, which never changes, exhibits dynamic phenomena and behavior in the presence of explicit information, and we are living proof of that.

Reference

The Implicit Information Hypothesis
https://timsamshuijzen.medium.com/the-implicit-information-hypothesis-5768341a68e3

Proof by diagonalization

$\Rightarrow$ prove things in a self-referential way

Cantor's diagonal argument

There exists no bijection between infinite sequences of 0's and 1's (binary sequences) and natural numbers. In other words, there is no way for us to enumerate all infinite binary sequences.

$$
\begin{array}{ll}
s_1 & = 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = 1\,1\,0\,1\,0\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,0\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,1\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,1\,1\,0\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,0\,1\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\ddots
\end{array}
$$

if s = 10111010011... $= s_i$, then $s_i[i] = 1$ or 0 ?

For any set X, there exists no bijection between X and the set of all subsets of X.

1. Assume such bijection exsits, and name it e
2. Define $S = \{x \in X | x \notin e(x)\}$.
   S is obviously a subset of X. Hence, there exists some $u \in X$ such that $e(u) = S$
3. If $u \in S \Rightarrow u \notin e(u) = S$
   If $u \notin S \Rightarrow u \in e(u) = S$
4. $u \in S$ or $u \notin S$?

Russell's paradox

$$R = \{S \ set | S \notin S\}$$

1. If $R \in R \Rightarrow$ by definition R, $R \notin R$
2. If $R \notin R \Rightarrow R \in R$

Solution : R cannot be a set

Halting Problem

No algorithm exists to decide whether a given program terminates

1. Assume there is such an algorithm A that takes an input (X, Y) and determines whether algorithm X ran on its input Y terminates
2. Define another algorithm B that takes an input W and does the following
   - if A ran on (W, W) returns True, then execute an infinite loop
   - if A ran on (W, W) returns False, then terminate the execution

Consider running A on input (B, B)

- if A ran on (B, B) returns True $\Rightarrow$ B halts on B $\Rightarrow$ A ran on (B, B) should return False
- if A ran on (B, B) returns False $\Rightarrow$ B doesn't halt on B $\Rightarrow$ A ran on (B, B) should return True

Gödel's first incompleteness theorem

In non-contradictory mathematics, based on a reasonably large amount of axioms, there will always be a sentence that cannot be proved or disproved.

Ex : (A) The sentence A cannot be proved true

Reference

The diagonalization argument
https://nseverkar.medium.com/the-diagonalization-argument-53dca529570e

Fermat's Little Theorem

Theorem

If p is a prime and a is any integer not divisible by p, then p divides $a^{p-1} - 1$

Proof using Multinomial Expansion.

$$a^p = (1 + 1 + 1 + ... + 1_a)^p = \sum_{k_1, k_2, ..., k_a} \binom{p}{k_1, k_2, ... k_a}$$

For those terms whose $k_1, k_2, ... k_a$ are all not p, they are multiples of p

There are exactly a terms that have one of k_i being p and those terms are all 1

We can conclude that $a^p - a$ is a multiple of p.

Since a is not divisible by p, $a^{p-1} - 1$ is a multiple of p □

Proof using Modular Arithmetic.

Consider a set of integer $\{a, 2a, 3a, ..., (p-1)a\}$. None of these numbers are congruent modulo p to any other, nor is any congruent to zero, i.e.,

$$r \times a \not\equiv s \times a \text{ (mod p)}, \ 1 \leq r < s \leq (p-1)$$
$$r \times a \not\equiv 0 \text{ (mod p)}, \ 1 \leq r \leq (p-1)$$

After dividing all integers in the set by p, we can get remainders exactly 1,2,...,(p-1). Hence, by modular arithmetic,

$$a \times 2a \times ... \times (p-1)a \equiv 1 \times 2 \times ... \times (p-1) \text{ (mod p)}$$
$$\Rightarrow a^{p-1}(p-1)! \equiv (p-1)! \text{ (mod p)}$$

Since (p-1)! is not divisible by p, $a^{p-1} \equiv 1$ (mod p) □

Reference

Fermat's Little Theorem
https://www.cantorsparadise.com/fermats-little-theorem-fbc88498d54e

The three 3s problem

3 3 3 = 1	3 3 3 = 6
3 3 3 = 2	3 3 3 = 7
3 3 3 = 3	3 3 3 = 8
3 3 3 = 4	3 3 3 = 9
3 3 3 = 5	3 3 3 = 10

⋆ 只能用 ()+-*/! 完成那十條等式

- 1 到 9 相對容易
  - (3!-3)/3=1
  - (3+3)/3=2
  - 3/3*3=3
  - 3+(3/3)=4
  - (3!/3)+3=5
  - 3!*3/3=6
  - 3!+(3/3)=7
  - 3!+3!/3=8
  - 3+3+3=9
- 但 10 要怎麼產生呢? → !3+!3+3!=10

  what is !3 ? → the subfactorial function

- The subfactorial function gives the number of derangements possible for a given set. (a derangement is a permutation of the elements of a set, such that no element appears in its original position.)
  - $!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n \binom{n}{n}(n-n)!$
    $= \sum_{k=0}^{n} \binom{n}{k}(-1)^k (n-k)!$
    $= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \Rightarrow \lim_{n\to\infty} \frac{!n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = e^{-1}$
  - $!n = n! - \sum_{k=1}^{n} \binom{n}{k} !(n-1)$
  - $!n = (n-1)(!(n-1) + !(n-2))$
- $10 = !3 + !3 + 3! = 2 + 2 + 6$

Reference

- How To Really Solve The Three 3s Problem?
  https://medium.com/street-science/how-to-really-solve-the-three-3s-problem-96d9246690bc
- Derangement
  https://en.wikipedia.org/wiki/Derangement

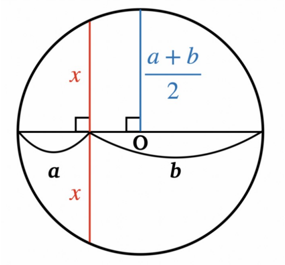
AM-GM Inequality

Theorem (AM-GM Inequality)

$\frac{a+b}{2} \geq \sqrt{ab}$, *with equality if and only if a=b*

$a * b = x * x \therefore x = \sqrt{ab}$

將紅線向右移，移到與直徑重合可證明等式



Reference

Visual Proof of Inequality of Arithmetic and Geometric Means (AM—GM Inequality): (a + b)/2 $\geq \sqrt{ab}$ https://medium.com/@satoshihgsn/visual-proof-of-inequality-of-arithmetic-and-geometric-means-am-gm-inequality-a-b-2-ab-d7ec78e05292

π

- $\pi \approx 22/7$
- π is a transcendental number
  - A rational number is one that can be expressed as the ratio of two integers
    An algebraic number is one that is the root of a finite polynomial with integer coefficients
    - ★ Every rational number is algebraic because it is the root of the equation $qx - p = 0$
      $\Rightarrow$ every transcendental number is irrational
    - ★ There are algebraic numbers that are not rational. Ex : $\sqrt{2}$
  - Countries can show off their technology to other countries by calculating as many digits of π as possible because this task requires a powerful computer.

- 利用圓的內接正多邊形與外接正多邊形，去得到圓周長的上下界。隨著正多邊形的邊數越來越大，上下界會越來越接近，透過此法，可以去逼近圓周長，進而逼近 π
- Borwein Integrals
  - $\int_0^\infty \frac{sin(x)}{x} dx = \frac{\pi}{2}$
  - $\int_0^\infty \frac{sin(x)}{x} \frac{sin(x/3)}{x/3} dx = \frac{\pi}{2}$
  - $\int_0^\infty \frac{sin(x)}{x} \frac{sin(x/3)}{x/3} \frac{sin(x/5)}{x/5} dx = \frac{\pi}{2}$
  - $\int_0^\infty \frac{sin(x)}{x} \frac{sin(x/3)}{x/3} \ldots \frac{sin(x/13)}{x/13} dx = \frac{\pi}{2}$
  - $\int_0^\infty \frac{sin(x)}{x} \frac{sin(x/3)}{x/3} \ldots \frac{sin(x/15)}{x/15} dx \neq \frac{\pi}{2}$
- $tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

- $\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{26390n+1103}{396^{4n}}$
- $e^{-\frac{\pi}{2}} = i^i$
- Monte Carlo method to find π :
  - 在一個 1×1 的正方形內，隨機產生坐標點，有些點距離正方形左下角距離大於 1，有些小於 1。當點數夠大時，那些距離正方形左下角距離小於等於 1 的點佔所有點的比例趨近爲 $\pi/4$
  - 做一個半徑爲 1 之圓，並在其外外接一邊長 2 之正方形。在正方形內隨機產生座標點，當點數夠大時那些落在圓內的點佔所有點的比例趨近爲 $\pi/4$
- Buffon's Needle :
  假設有一張等間隔的格線紙，以及一個長度與格線紙間隔寬相同的針。將針隨意丟到該格線紙上，令 n 爲總共丟的針個數，而 c 爲有跨過任何一條線的針個數。若 n 夠大，2n/c 會趨近於 π

- 一條彎曲河流之總長爲 L，連接河首河尾之直線長爲 ℓ，sinuosity 定義爲 $\frac{L}{\ell}$ $\rightarrow$ 一個衡量河彎曲程度的指標

  ⋆ It has been proved that the average sinuosity of rivers around the world is π

- The Pi day：每年的 3/14
  - 愛因斯坦生於 1879 年 3/14

Reference

20+ Fascinating Beauties That Makes Pi(π) Unique
https://medium.com/however-mathematics/20-fascinating-beauties-that-makes-pi-%CF%80-unique-e78283a60596

e

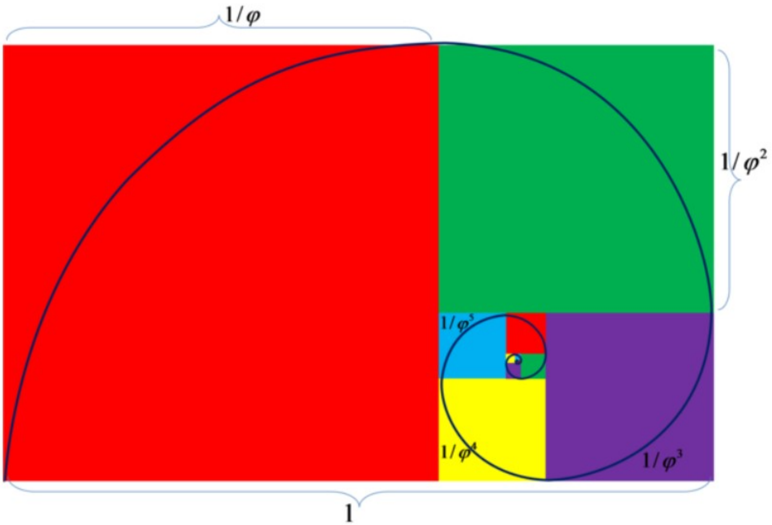
- $\lim_{n\to\infty}(1+\frac{1}{n})^n = e \Rightarrow$ e 代表了本利和的變化速度
- $\frac{d(e^x)}{dx} = e^x \Rightarrow e^x$ 在任何點的變化速率皆爲在該點的取值
- $\int_{-\infty}^{t} e^x dx = e^t \Rightarrow e^x$ 從負無窮大到任何點的面積恰爲在該點的取值

Reference

e —The Euler Number
https://medium.com/geekculture/e-the-euler-number-ef6d114da1d9

$$\varphi$$

- the most beautiful number in mathematics
- the golden ratio, the divine proportion
- $\varphi = \frac{1+\sqrt{5}}{2} = 1.68...$
- Fibonacci sequence：1, 1, 2, 3, 5, 8, 13, 21, ...
  $\rightarrow$ 將每一項除以前一項，得到之比值會收斂至 φ
- 蜂群中，母蜂的個數除以公蜂的個數爲 φ
- 肩膀到手指尖的長度除以手肘到手指間的長度爲 φ
- 髖部到腳趾尖的長度除以膝蓋到腳趾尖的長度爲 φ



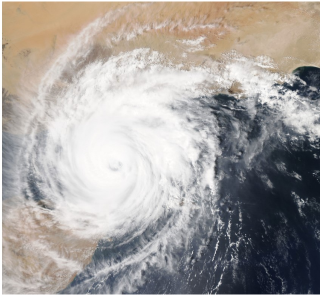
$1/\varphi$
$1/\varphi^2$
$1/\varphi^5$
$1/\varphi^4$
$1/\varphi^3$
1

(a) sunflower

(b) nautilus

(c) Milky Way galaxy

(d) 蕨類



(a) hurricane

(b) 松果

(c) 多葉蘆薈

Reference

〈黃金比例 2〉: 大自然的黃金比例螺線
https://www.taiwannews.com.tw/ch/news/3472903

The Most Beautiful Number in Mathematics
https://medium.com/nerd-for-tech/the-most-beautiful-number-in-mathematics-ff10cc424faa