Interpolation for One-Dimensional Data

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- Objective: create and predict extra reasonable data between existing data
- Applications
  - Fill gaps in time series data because recorded time series data may have missing data or are nonuniformly sampled.
  - Upsample for increasing the resolution of time series data
  - Data augmentation for better data analysis, numerical computing, machine learning (avoid overfitting and boost generalization).
  - Perform smoothing between discrete data points by producing extra points.
  - Estimate and approximate the unknown data generating function
  - Generate continuous curves for data fitting to enable data explanation.

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Sinc Interpolation Method

- The sinc interpolation method, also known as bandlimited interpolation, is a mathematical technique used to reconstruct a continuous signal from its discrete samples.

- It is closely tied to the Nyquist-Shannon sampling theorem, which states that a bandlimited signal can be perfectly reconstructed from its samples if the sampling rate is at least twice the highest frequency present in the signal (the Nyquist rate).

Sinc Interpolation Method

The sinc interpolation formula for reconstructing a continuous-time signal $x(t)$ from its discrete samples $x[n]$ is

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \, \mathrm{sinc}\left(\frac{t - nT}{T}\right)$$

where

- $x[n]$ are the sampled values of the signal.
- T is the sampling interval.
- $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

Sinc Interpolation Method

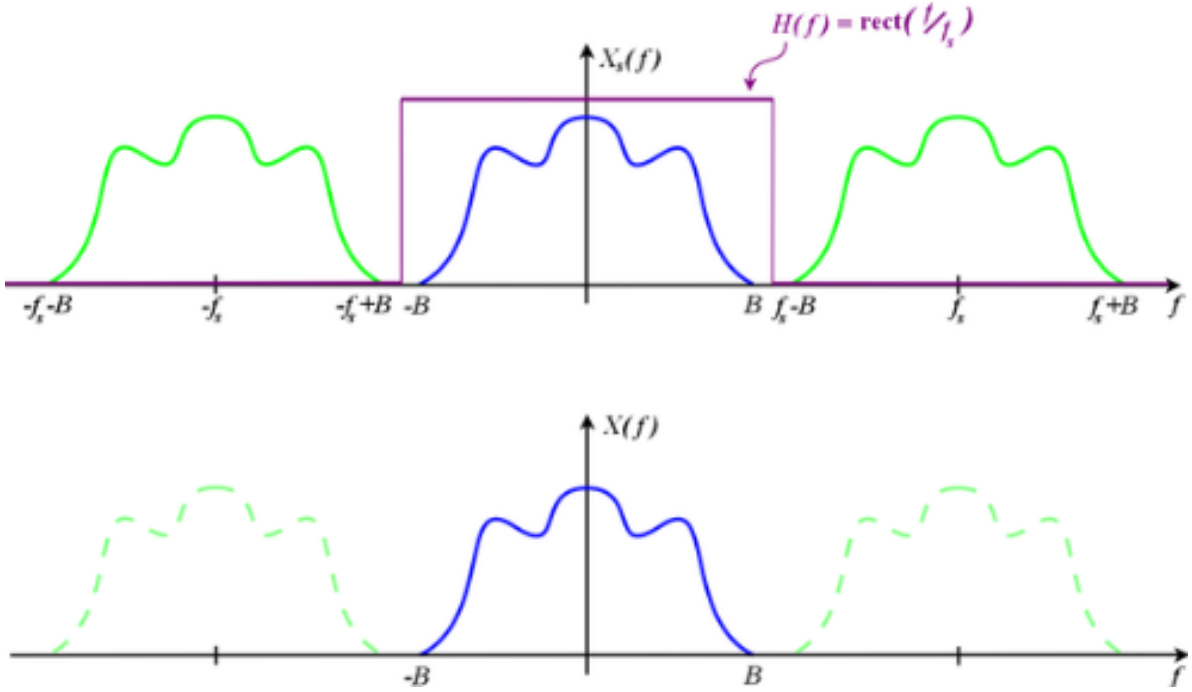


Figure: Sinc Interpolation: Reconstruction by An Ideal Lowpass Filter

Polynomial Interpolation Methods

1. Lagrange Interpolation:
   Given a set of $n+1$ timestamps $\{x_0, x_1, ..., x_n\}$, create a set of Lagrange polynomials $\{\ell_0(x), \ell_1(x), ..., \ell_n(x)\}$ each of degree n:

$$\ell_j(x) = \frac{x - x_0}{x_j - x_0} ... \frac{x - x_{j-1}}{x_j - x_{j-1}} \frac{x - x_{j+1}}{x_j - x_{j+1}} ... \frac{x - x_n}{x_j - x_n}$$

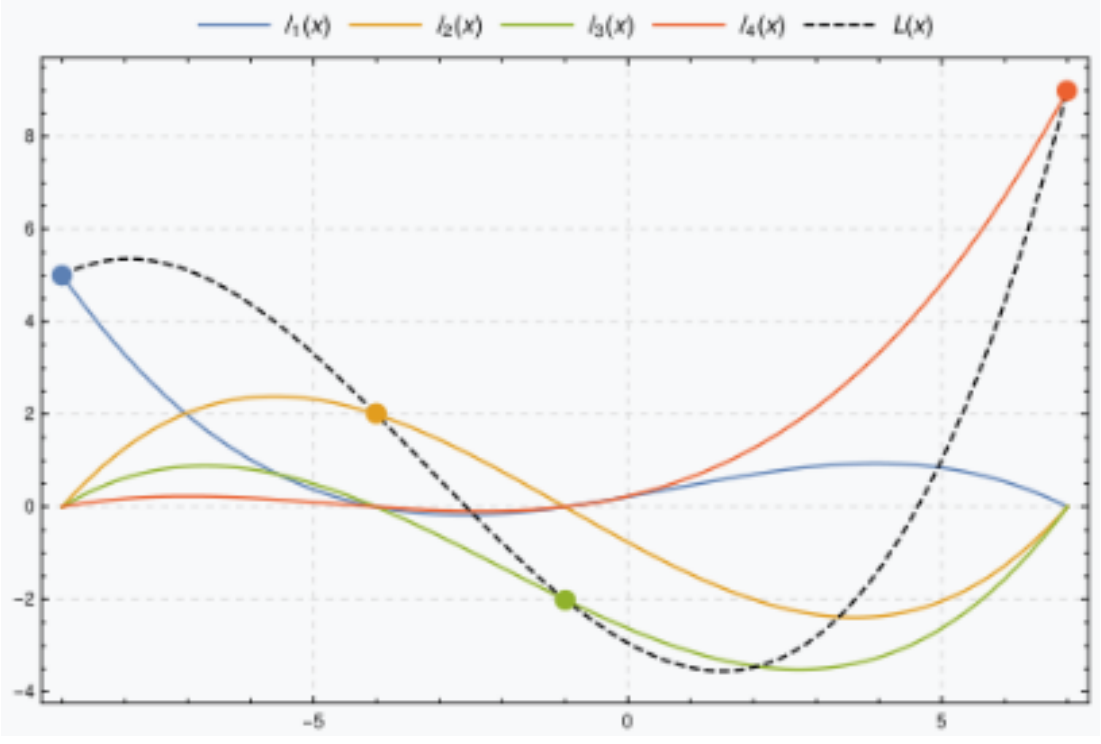
   Thus, $\ell_j(x_m) = 0$ if $m \neq j$ and $\ell_j(x_j) = 1$. The Lagrange interpolation is performed through

$$L(x) = \sum_{j=0}^{n} y_j \ell_j(x)$$

   where $L(x)$ passes through $\{(x_0, y_0), ..., (x_n, y_n)\}$

Polynomial Interpolation Methods

1. Lagrange Interpolation:



Polynomial Interpolation Methods

2. Barycentric Rational Interpolation by Floater and Hormann: The Floater-Hormann Interpolant $R(x)$

$$R(x) = \frac{\sum_{i=0}^{n} w_i \frac{y_i}{x-x_i}}{\sum_{i=0}^{n} w_i \frac{1}{x-x_i}}$$

where w_i are weights that depend on the choice of a degree parameter d. They are calculated as

$$w_i = \sum_{k \in J_i} (-1)^k \prod_{j=k, j \neq i}^{k+d} \frac{1}{x_i - x_j}$$

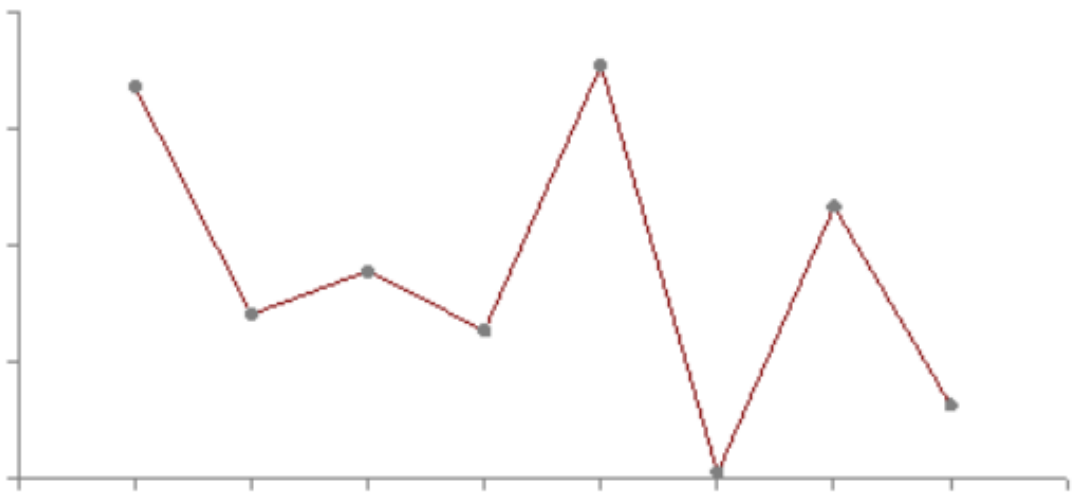
$$J_i = \{k \in \{0, 1, ..., n-d\} : i - d \leq k \leq i\}$$

Polynomial Interpolation Methods

2. Barycentric Rational Interpolation by Floater and Hormann:
   - The use of rational functions can mitigate oscillations that may occur with high-degree polynomial interpolation $\rightarrow$ avoid the Runge's phenomenon
   - The Barycentric form is computationally efficient and numerically stable
   - d controls the degree of the local polynomial used in the rational approximation. The choice of d influences the smoothness and accuracy of the interpolation.

Piecewise Polynomial Interpolation Methods

1. Linear Interpolation: join every two neighboring points by a straight line
   - Piecewise linear polynomial causes discontinuities at each point.



Piecewise Polynomial Interpolation Methods

2. Cubic Interpolation: constructs a cubic polynomial

$$f(x) = a_i(x - x_i)^2 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

for each interval $[x_i, x_{i+1}]$ to interpolate between data points, where a_i, b_i, c_i and d_i are determined by

1) Matching function values at endpoints:

$$f(x_i) = y_i, f(x_{i+1}) = y_{i+1}$$

2) Matching first derivatives at endpoints:

$$f'(x_i) = y_i', f'(x_{i+1}) = y_{i+1}'$$

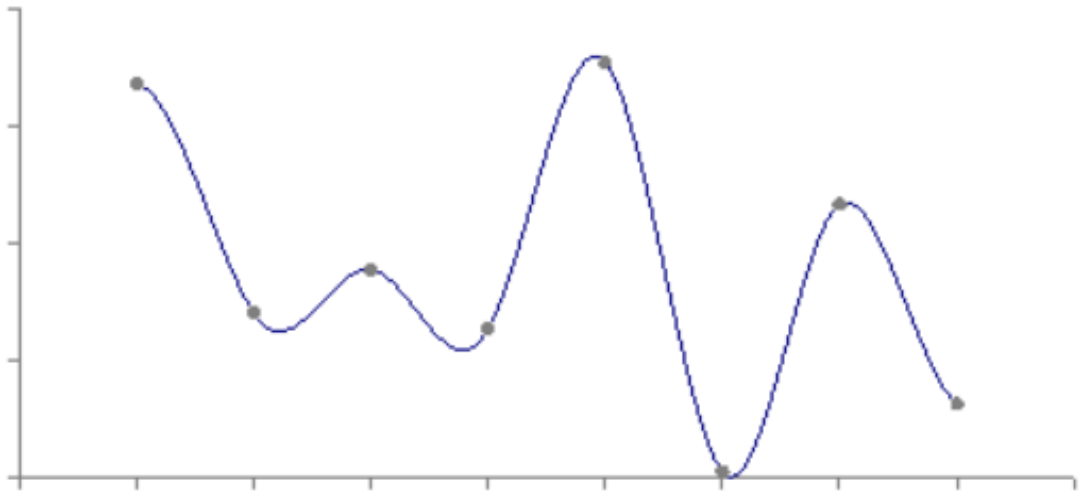
Piecewise Polynomial Interpolation Methods

2. Cubic Interpolation:

   - Cubic interpolation can ensure that the function and its first derivative are continuous at the interpolation points.
   - If we don't know the derivative of the function, we could simply use derivative 0 at every point.
   - We can obtain smoother curves when we use the slope of a line between the previous and the next point as the derivative at a point.
     $\rightarrow$ the resulting polynomial is called a Catmull-Rom spline

Piecewise Polynomial Interpolation Methods

2. Cubic Interpolation:



Piecewise Polynomial Interpolation Methods

3. Cubic Hermite Interpolation: uses both the function values y_i and derivatives y_i' at the data points. The interpolating cubic Hermite polynomial is

$$f(x) = h_0(t)y_i + h_1(t)y_{i+1} + h_2(t)y_i' + h_3(t)y_{i+1}'$$

where

$$t = \frac{x - x_i}{x_{i+1} - x_i}$$

and the Hermite basis functions are

$$h_0(t) = 2t^3 - 3t^2 + 1, h_1(t) = -2t^3 + 3t^2$$
$$h_2(t) = t^3 - 2t^2 + t, h_3(t) = t^3 - t^2$$

Piecewise Polynomial Interpolation Methods

4. Spline Interpolation: constructs a cubic polynomial

$$S_i(x) = a_i(x - x_i)^2 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

for each interval $[x_i, x_{i+1}]$ to interpolate between data points, where a_i, b_i, c_i and d_i are determined by

1) Matching function values at endpoints:

$$S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1}$$

2) Continuity of first and second derivatives:

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$

Piecewise Polynomial Interpolation Methods

4. Spline Interpolation: constructs a cubic polynomial

$$S_i(x) = a_i(x - x_i)^2 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

for each interval $[x_i, x_{i+1}]$ to interpolate between data points, where a_i, b_i, c_i and d_i are determined by

   3) Boundary conditions:

      - Natural spline: $S_0''(x_0) = S_{n-1}''(x_n) = 0$
      - Clamped spline: Specify $S_0'(x_0)$ and $S_{n-1}'(x_n)$

Piecewise Polynomial Interpolation Methods

5. B-Spline Interpolation: uses a linear combination of basis splines, defined by control points and the knot vector. The B-spline is

$$f(x) = \sum_{i=0}^{m} B_{i,k}(x) P_i$$

where

   1. P_i: control points
   2. $B_{i,k}(x)$: B-spline basis functions of degree k

Piecewise Polynomial Interpolation Methods

5. B-Spline Interpolation: uses a linear combination of basis splines, defined by control points and the knot vector. The B-spline is

$$f(x) = \sum_{i=0}^{m} B_{i,k}(x) P_i$$

$B_{i,k}(x)$ are computed recursively:

$$B_{i,0}(x) = \begin{cases} 1 & t_i \leq x < t_{i+1} \\ 0 & \textit{otherwise} \end{cases}$$

$$B_{i,k}(x) = \frac{x - t_i}{t_{i+k} - t_i} B_{i,k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(x)$$

where $t = \{t_0, t_1, ..., t_m\}$ is the knot vector determining where and how the basis functions are defined.

Piecewise Polynomial Interpolation Methods

5. B-Spline Interpolation:
   - If $t_{i+k} - t_i = 0$, then $\frac{x-t_i}{t_{i+k}-t_i}$ is treated as 0 ; If $t_{i+k+1} - t_{i+1} = 0$, then $\frac{t_{i+k+1}-x}{t_{i+k+1}-t_{i+1}}$ is treated as 0.
   - If the original knot vector is$\{t_0, t_1, ..., t_m\}$, we need to extend it by repeating the first and last knots $k+1$ times. For instance, for a cubic B-spline (k=3), the extended knot vector would be
     $$\{t_0, t_0, t_0, t_0, t_1, ..., t_{m-1}, t_m, t_m, t_m, t_m\}$$
     This ensures that
     - $B_{i,k}(x)$ at the boundaries (where i=m) is defined properly
     - $B_{m,k}(x)$ and $B_{0,k}(x)$ transition smoothly to 0 outside the domain of interest
   - The control points $\{P_0, P_1, ..., P_m\}$ can be determined by minimizing $\sum_{j=0}^{n}(y_j - \sum_{i=0}^{m} B_{i,k}P_i)^2$

Piecewise Polynomial Interpolation Methods

6. Bézier Spline Interpolation: Bézier splines are parameterized curves defined by control points. A Bézier curve of degree k is

$$B(t) = \sum_{i=0}^{k} \binom{k}{i} (1-t)^{n-i} t^i P_i$$

where P_i are the control points.

Piecewise Polynomial Interpolation Methods

6. Bézier Spline Interpolation:
   1) Single Bézier Curve
      - If a single Bézier curve is used, it is defined by a single polynomial of degree k and passes through or approximates the data points. This approach does not divide the curve into segments, so it is not piecewise.
   2) Composite Bézier Splines
      - A composite Bézier spline is a piecewise polynomial interpolation method. It connects multiple Bézier curves end-to-end to form a larger curve that passes through or approximates data points.
      - Each segment is a separate Bézier curve (defined by its control points), and smoothness at the connections (e.g. first-derivative and second-derivative continuity) is achieved by appropriately choosing the control points.

Regression-Based Methods

1. Kriging
   - The method is named after the South African mining engineer Danie Krige, who first introduced the technique in the 1950s for mining applications.
   - It is a statistical technique used for spatial data interpolation, commonly applied in fields such as geophysics, geostatistics, environmental modeling, and even machine learning.
   - It is particularly valuable when dealing with data that has spatial or temporal correlation, meaning that the values of a variable at nearby locations are likely to be similar.
     $\rightarrow$ The primary goal is to predict the value of a field (or signal) at an unmeasured location, based on the known values at surrounding locations.

Regression-Based Methods

1. Kriging

   The Kriging estimator at location x_0 is written as

$$\hat{Z}(x_0) = m + \sum_{i=0}^{n} \lambda_i Z(x_i)$$

   where $\hat{Z}(x_0)$ is the predicted value at location x_0, m is the known stationary average, n is the number of known data points, $Z(x_i)$ assumes to be stochastic with a zero mean, constant variance, and a non-constant covariance, and λ_i are the weights associated with each data point $Z(x_i)$.

   - The weights are obtained by solving a system of equations that minimizes the prediction variance, considering the spatial correlation between points as defined by the variogram.
   - Kriging can be seen a stochastic regression method.

Regression-Based Methods

2. Kernel Methods

   - Kernel interpolation methods involve using a kernel function to compute the weight for each known data point, which is then used to estimate the value at a new, unobserved point.
   - These methods often rely on the Mercer's theorem, which states that any positive semidefinite function can be represented as an inner product in a higher-dimensional feature space.
   - The kernel acts as the inner product in the high-dimensional feature space.

Regression-Based Methods

2. Kernel Methods : The general framework of kernel interpolation is

$$f(x) = \sum_{i=1}^{n} \alpha_i k(x, x_i)$$

   - $f(x)$ is the estimated value at the point x
   - x_i are known data points
   - α_i are weights associated with the data points
   - $k(x, x_i)$ is the kernel function that measures the similarity between the point x and each x_i.

   The weights α_i are usually determined by solving a system of equations based on the interpolation conditions, which can vary depending on the type of kernel method being used.

Regression-Based Methods

2. Kernel Methods
   - Common kernel functions
     - Linear kernel: $k(x, x') = x^T x'$
       - ⋆ This kernel is used when the data is assumed to be linearly related
     - Polynomial kernel: $k(x, x') = (x^T x' + c)^d$
       - ⋆ A polynomial kernel introduces polynomial relationships between points.
     - Sigmoid kernel: $k(x, x') = \tanh(\alpha x^T x' + c)$
       - ⋆ Used for modeling more complex relationships, and it's often employed in SVM-based kernel interpolation methods.

Regression-Based Methods

2. Kernel Methods
   - Common kernel functions
     - Gaussian (RBF) kernel: $k(x, x') = \exp(-\frac{\|x-x'\|_2^2}{2\sigma^2})$
       - ⋆ The Gaussian kernel (also called the Radial Basis Function (RBF) kernel) is widely used because it can model non-linear relationships and is effective for interpolation in high-dimensional spaces.
     - Laplacian kernel: $k(x, x') = \exp(-\frac{\|x-x'\|}{\sigma})$

       ⋆ This kernel is similar to the Gaussian kernel but decays more slowly, allowing it to model different types of relationships.

Regression-Based Methods

2. Kernel Methods: Kernel Ridge Regression (KRR) is one of the most popular kernel-based interpolation techniques, combining the idea of ridge regression (a form of regularized linear regression) with the kernel trick
   - The weights α_i are determined by minimizing the cost function

$$J(\alpha) = \sum_{i=1}^{n} (y_i - f(x_i))^2 + \lambda \sum_{i=1}^{n} \alpha_i^2$$

The system can be efficiently solved by

$$\alpha = (K + \lambda I)^{-1} y$$

where K is the kernel matrix, whose elements are $k(x_i, x_j)$.

Disadvantages of Famous Methods

- Sinc interpolation
  - The disadvantage of perfect reconstruction is the inability to deal with noisy cases.
  - Sinc interpolation requires the input data to be uniformly sampled.
  - The sampling frequency should be beyond the Nyquist rate.
- Polynomial interpolation and piecewise polynomial interpolation
  - Runge's phenomenon occurs with high-degree polynomial interpolation.
  - The estimated curve passes through all the given points.
    $\rightarrow$ the interpolation result is poor once the data is contaminated with noise.

Disadvantages of Famous Methods

- Regression-Based Methods
  - Statistical properties of the underlying stochastic process should be known or assumed for Kriging.
  - Adoption of complex kernels may lead to overfitting for the kernel methods.
  - We may want to use more flexible basis functions beyond the random process, or kernels.

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Intuition and Methodology

1. For a bandlimited function $f(t)$, we can express it using a complete orthogonal bandlimited function set $\{b_i(t)\}$

$$f(t) = \sum_{i=1}^{\infty} a_i b_i(t), \quad t \in (-\infty, \infty)$$

2. $f(t) = \sum_{i=1}^{\infty} a_i b_i(t)$ for $t \in (-T/2, T/2)$ also holds.

$$\Rightarrow \|f(t)\|_{(-T/2,T/2)} = \sum_{i=1}^{\infty} |a_i| \|b_i(t)\|_{(-T/2,T/2)}$$

3. Practically, we can only use finite number of basis functions

$$\|f(t)\|_{(-T/2,T/2)} \geq \sum_{i=1}^{m} |a_i| \|b_i(t)\|_{(-T/2,T/2)}$$

Intuition and Methodology

4. Among all complete orthogonal bandlimited function sets, the prolate spheroidal wave functions (PSWFs) $\{\psi_i(c,t)\}$ concentrates most energy in the interval $(-T/2, T/2)$, where $c = \Omega T/2$ and Ω is the half bandwidth.

$$\Rightarrow \sum_{i=1}^{m} |a_i| \|\psi_i(c,t)\|_{(-T/2,T/2)} = \max_{\{b_i(t)\}} \sum_{i=1}^{m} |a_i| \|b_i(t)\|_{(-T/2,T/2)}$$

5. We can even nonuniformly divide the bandwidth into several dominant subbands, and use a set of PSWFs $\{\psi_i(c_j,t)\}$ to approximate each subband.
   - The number of required basis functions decreases
   - More accurate approximation can be achieved
     - Some energies that leak off from each subband are still within the whole band.

Prolate Spheroidal Wave Functions

PSWFs are denoted by $\psi_i(c,t), i = 0, 1, \ldots,$ where $c = \Omega T/2$ and its Fourier transform concentrates in the band of $\omega \in (-\Omega, \Omega)$. PSWFs satisfy the following properties:

1. Eigenfunction Property: All the PSWFs satisfy

$$\lambda_i(c)\,\psi_i(c,t) = \int_{-T/2}^{T/2} \frac{\sin\Omega(t-s)}{\pi(t-s)}\psi_i(c,s)\,ds$$

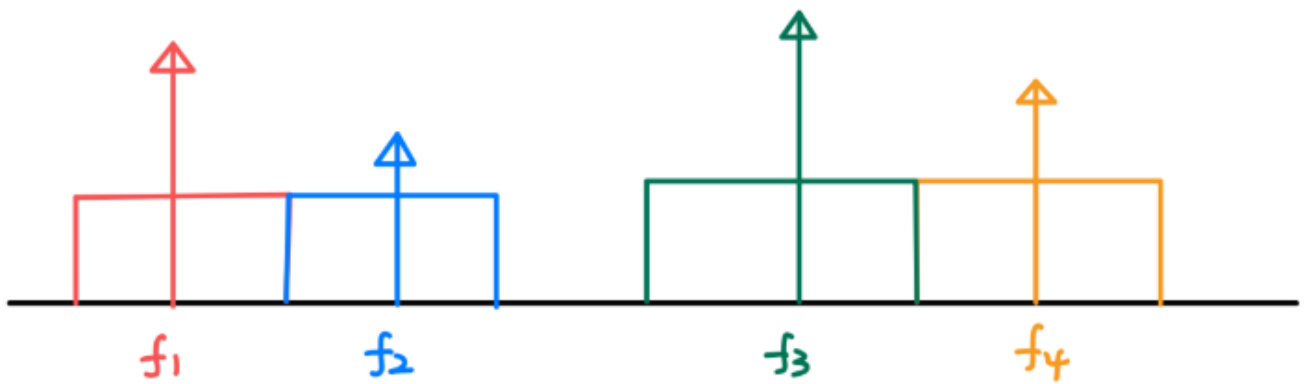
   Furthermore, $\lambda_i(c)$ satisfies $1 > \lambda_0(c) > \lambda_1(c) > \cdots > 0$ and for a given c, $\lambda_i(c)$ falls off to zero rapidly with i once i exceeds $(2/\pi)\,c$.

2. Orthogonality: $\{\psi_i(c,t), i = 0, 1, 2, \cdots\}$ forms an orthogonal and complete set in $\mathcal{L}^2_{T/2}$ while it forms an orthonormal and complete set in $\mathcal{B}$.

NUBD-PSWF

The proposed non-uniform band divided PSWF (NUBD-PSWF) is introduced as follows.

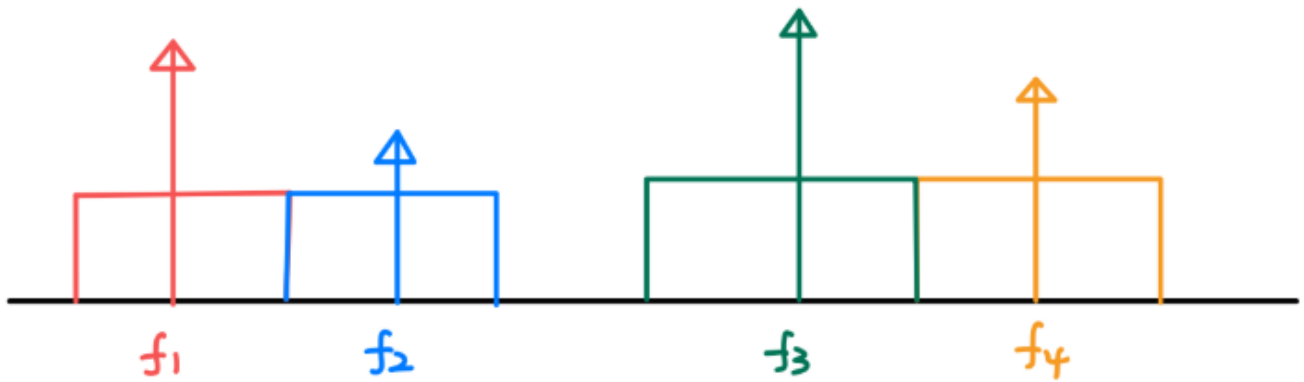
- Given a signal, we compute its discrete Fourier transform and pick out several frequencies with large absolute values as dominant frequencies, e.g. f_1, f_2, f_3, and f_4.



NUBD-PSWF

The proposed non-uniform band divided PSWF (NUBD-PSWF) is introduced as follows.

- Select appropriate bandwidths for each subband (e.g. the red, blue, green, and orange rectangles) so that subbands do not overlap with each other, but still manage to cover the whole band.



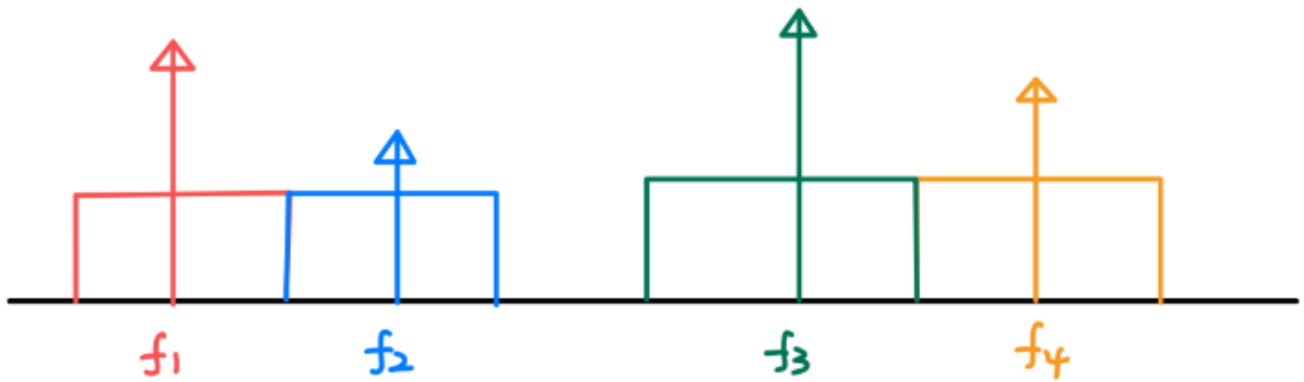
NUBD-PSWF

The proposed non-uniform band divided PSWF (NUBD-PSWF) is introduced as follows.

- Modulate each PSWFs function set to appropriate subband

$$\{e^{j2\pi f_i t}\psi_j(c_i, t)\}$$

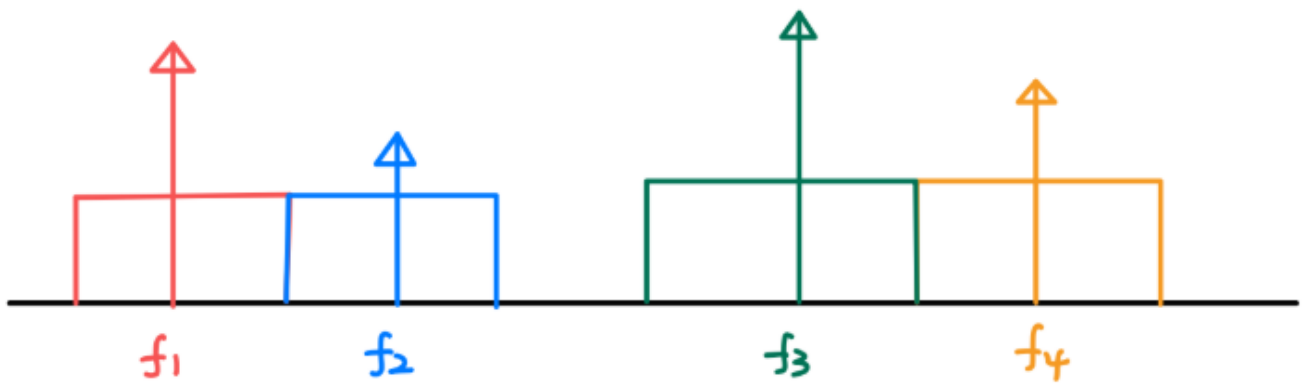
where $c_i = \Omega_i T/2$, and Ω_i is the bandwidth of the subband associated with f_i.



NUBD-PSWF

The proposed non-uniform band divided PSWF (NUBD-PSWF) is introduced as follows.

- We can collect these functions to serve as basis functions, and perform linear regression to approximate the input signal generating mechanism, and further do interpolation.



- NUBD-PSWF is a regression-based method, which uses modulated PSWFs as basis functions.
- Sparsely choosing dominant frequencies enables the NUBD-PSWF method to address noisy cases, and to avoid the over-fitting problem.
- Based on the idea of identifying dominant frequencies, we proceed on following works to improve interpolation results.
  - Use the sparse approximation concept and optimization techniques to identify dominant frequencies
    - The input data can be nonuniformly sampled.
    - The sampling frequency can below the Nyquist rate.
  - Use advanced time-frequency analysis to capture time-frequency components of the input signal, and use generalized chirp modulation to modulate PSWFs.

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- Nyquist-Shannon sampling theorem https://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem
- Lagrange polynomial https://en.wikipedia.org/wiki/Lagrange_polynomial
- M. S. Floater and K. Hormann. "Barycentric rational interpolation with no poles and high rates of approximation," Numerische Mathematik, vol. 107, pp. 315-331, 2007.
- M. Lepot, J. B. Aubin, and F. H. Clemens, "Interpolation in time series: An introductive overview of existing methods, their performance criteria and uncertainty assessment," Water, vol. 9, issue 10, article 796, 2017.
- P. Bourke, "Interpolation methods," Miscellaneous: projection, modelling, rendering, vol. 1, no. 10, 1999.

- Cubic interpolation
  https://www.paulinternet.nl/?page=bicubic
- Cubic Hermite spline
  https://en.wikipedia.org/wiki/Cubic_Hermite_spline
- Spline interpolation
  https://en.wikipedia.org/wiki/Spline_interpolation
- B-spline
  https://en.wikipedia.org/wiki/B-spline
- Bézier curve
  https://en.wikipedia.org/wiki/B%C3%A9zier_curve
- C.-J. Shih and J.-J. Ding, "Interpolation and extrapolation by prolate spheroidal wave functions using nonuniform division and generalized chirp modulation," in IEEE International Symposium on Circuits and Systems (ISCAS). IEEE, 2024, pp. 1-5.