Synchrosqueezing-Based Short-Time Fractional Fourier Transform

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Outline

- Review Synchrosqueezing Transform(SST) and Short-Time Fractional Fourier Transform(STFrFT)
- SST-Based STFrFT
- Conclusion

Synchrosqueezing Transform (FSST)

The modified STFT

$$STFT_s^g(t,\omega) = \int s(\tau)g(\tau - t)e^{-i\omega(\tau-t)}d\tau$$

For the mono-component signal model

$$s(t) = A(t)e^{i\phi(t)}, \text{satisfy } |A'(t)| \leq \varepsilon, \phi''(t) \leq \varepsilon$$

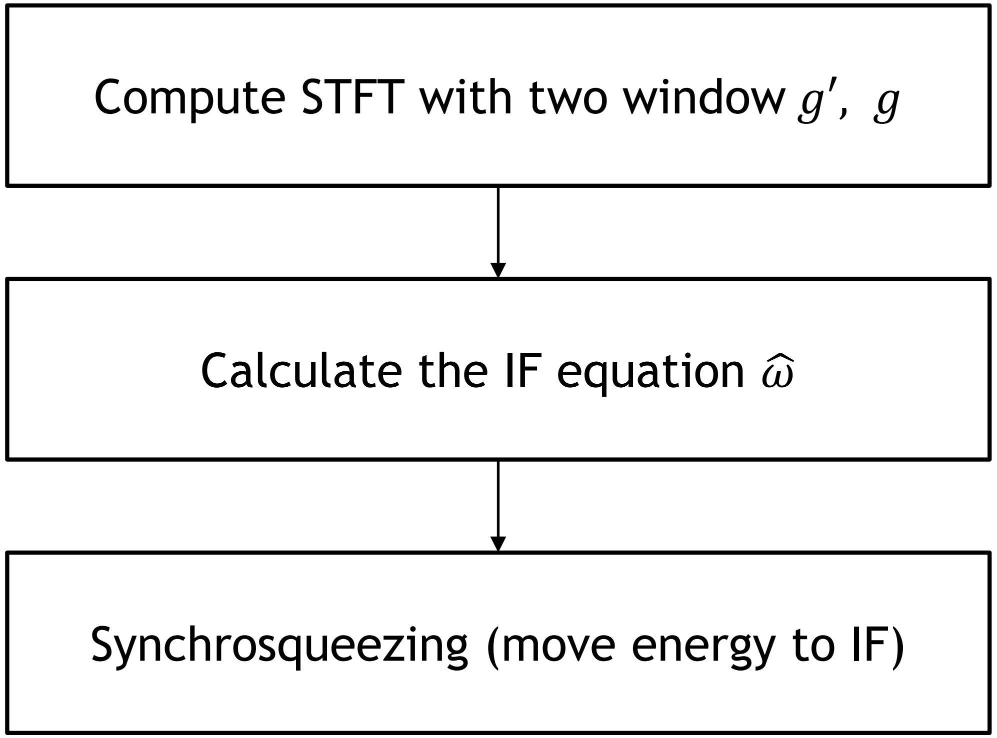
Instantaneous frequency (IF), (function of time on time-frequency plane), can be estimated by

$$\widehat{\omega}(t,\eta) = \eta - \Im\left\{\frac{STFT_s^{g'}(t,\omega)}{iSTFT_s^g(t,\omega)}\right\} = \phi'$$

Synchrosqueezing Transform (SST)

$$SST(t,\omega) = \int STFT(t,\eta)\delta\big(\omega - \widehat{\omega}(t,\eta)\big)d\eta$$

Synchrosqueezing Transform based on STFT (FSST)



Goal

- Signal is weak phase modulation.
- Since Short time Fourier transform ➔ Short time fractional Fourier transform. General SST based on fractional Fourier transform.

FrFT and Short Time FrFT

- Fractional Fourier Transform(FrFT)

$$FrFT_{\alpha}(u) = \mathcal{F}_{\alpha}\{s(t)\} = \int s(t)K_{\alpha}(t,u)dt$$

- Short-Time Fractional Fourier Transform(STFrFT)

$$STFrFT_{s,\alpha}^{g}(t,u) = \int s(\tau)g(\tau - t)K_{\alpha}(\tau,u)d\tau$$

where the kernel function is defined as

$$K_{\alpha}(\tau,u) = \begin{cases} B_{\alpha}e^{i\frac{t^2}{2}cot\alpha - iutcsc\alpha} & ,\alpha \neq k\pi \\ \delta(u-t), & \alpha = 2k\pi \\ \delta(u+t), & \alpha = (2k+1)\pi \end{cases}$$

δ(·) is the Dirac Delta function, t and u denote time and fractional frequency, respectively.

Fractional Synchrosqueezing Transform

- For the mono-component signal model (by Taylor expansion)

$$s(\tau) = A(t)e^{i\left(\phi(t)+\phi'(\tau-t)+\phi''(\tau-t)\right)}$$

- IF can be estimated by

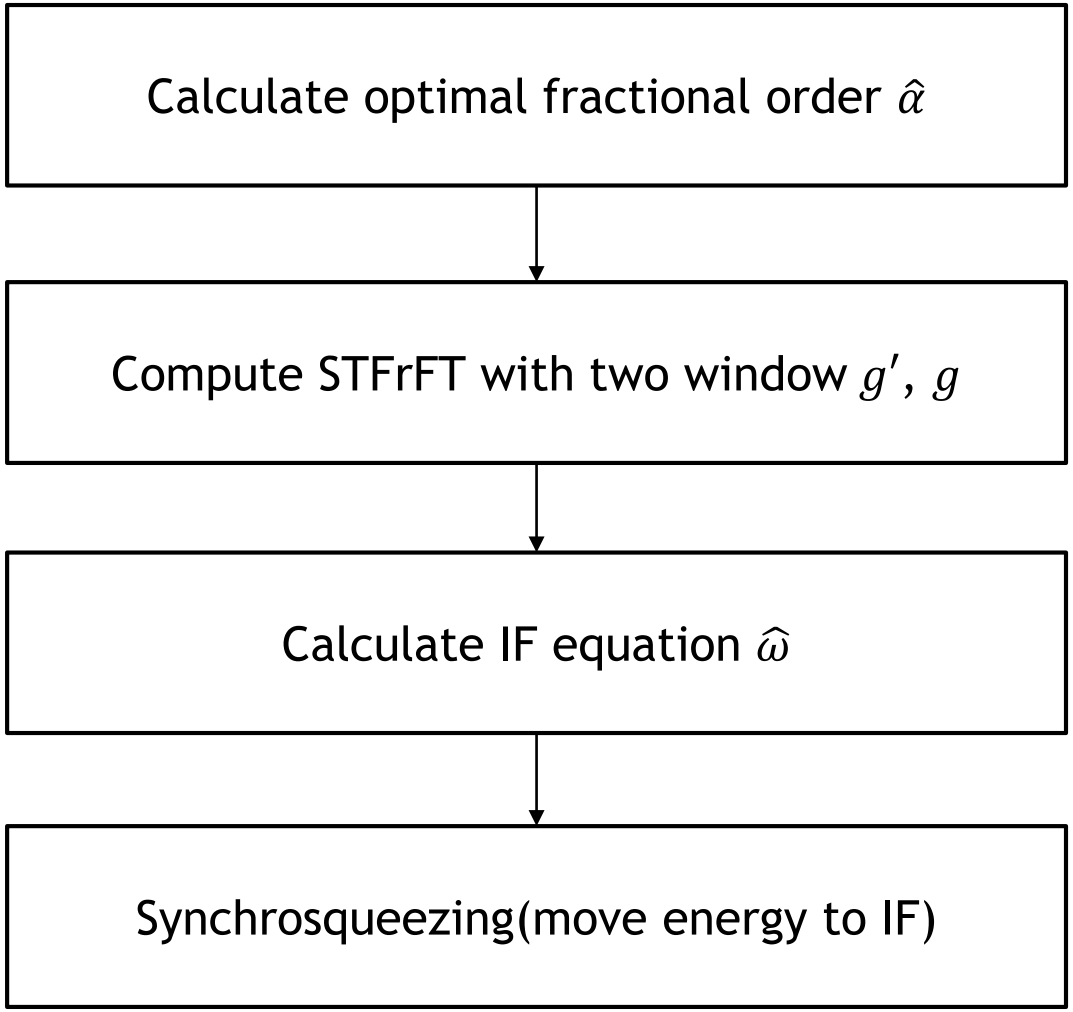
$$\hat{\omega}(t,\eta) = \Re\left\{u\csc\hat{\alpha} - \frac{STFrFT_{s,\hat{\alpha}}^{g'}(t,u)}{iSTFrFT_{s,\hat{\alpha}}^{g}(t,u)}\right\} = \Re\left\{\omega - \frac{STFrFT_{s,\hat{\alpha}}^{g'}(t,\omega\cdot\sin\hat{\alpha})}{iSTFrFT_{s,\hat{\alpha}}^{g}(t,\omega\cdot\sin\hat{\alpha})}\right\} = \phi'$$

where optimal fractional order $\hat{\alpha} = arccot(-\phi'')$.

- Fractional Synchrosqueezing Transform (FrSST)

$$FrSST(t,\eta) = \int STFrFT_{s,\hat{\alpha}}^{g'}(t,\omega\cdot\sin\hat{\alpha})\delta\left(\eta - \hat{\omega}(t,\omega)\right)d\omega$$

Fractional Synchrosqueezing Transform



Fractional Synchrosqueezing Transform

- For the multi-component signal model

$$s(\tau) = \sum_{k=1}^{K} s_k = A_k(t)e^{i\phi_k(t)}$$

- Short Time FrFT

$$STFrFT_{s,\alpha}^{g}(t,u) = \sum_{k=1}^{K} STFrFT_{s_k,\alpha}^{g}(t,u) = \underbrace{\sum_{l=1}^{L} STFrFT_{s_{k_l},\alpha}^{g}(t,u)}_{\text{same chirp rate } \phi''_{k_l}} + \underbrace{\sum_{k\neq k_1,\cdots,k_L} STFrFT_{s_k,\alpha}^{g}(t,u)}_{\text{otherwise}}$$

- Optimal fractional order $\hat{\alpha}_0 = arccot\left(-\phi''_{k_l}\right)$.
- Approximation

$$STFrFT_{s,\hat{\alpha}_0}^{g}(t,u) \approx \sum_{l=1}^{L} STFrFT_{s_{k_l},\hat{\alpha}_0}^{g}(t,u)$$

Fractional Synchrosqueezing Transform

- For each chirp rate，IF can be estimated by

$$\widehat{\omega}(t,\omega;\hat{\alpha}_0) = \Re\left\{\omega - \frac{\sum_{l=1}^{L} STFrFT^{g'}_{s_{k_l},\hat{\alpha}_0}\left(t,\omega\cdot sin\widehat{\hat{\alpha}_0}\right)}{i\sum_{l=1}^{L} STFrFT^{g}_{s_{k_l},\hat{\alpha}_0}\left(t,\omega\cdot sin\hat{\alpha}_0\right)}\right\}$$

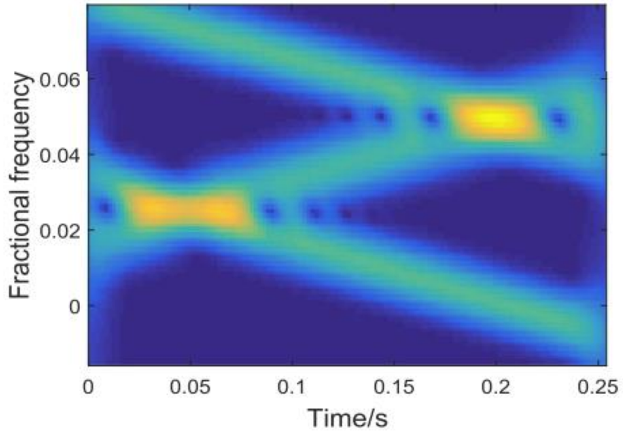
- IF can be estimated by summing up

$$\widehat{\omega}(t,\omega) = \sum_{\hat{\alpha}} \widehat{\omega}(t,\omega;\hat{\alpha}_0) = \sum_{\hat{\alpha}} \Re\left\{\omega - \frac{STFrFT^{g'}_{s,\hat{\alpha}}(t,\omega\cdot sin\hat{\alpha})}{iSTFrFT^{g}_{s,\hat{\alpha}}(t,\omega\cdot sin\hat{\alpha})}\right\}$$

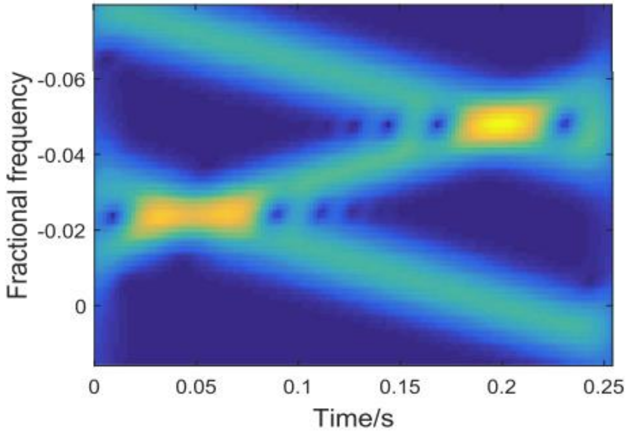
- Synchrosqueezing Transform

$$FrSST(t,\eta) = \sum_{\hat{\alpha}}\sum_{l=1}^{L} FrSST_{s_{k_l}}(t,\eta)$$

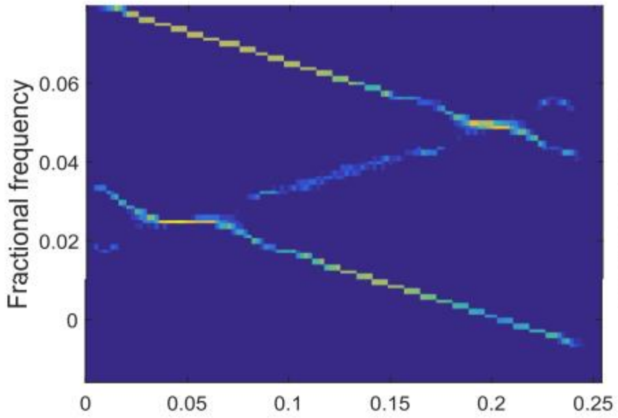
STFrFT α_1



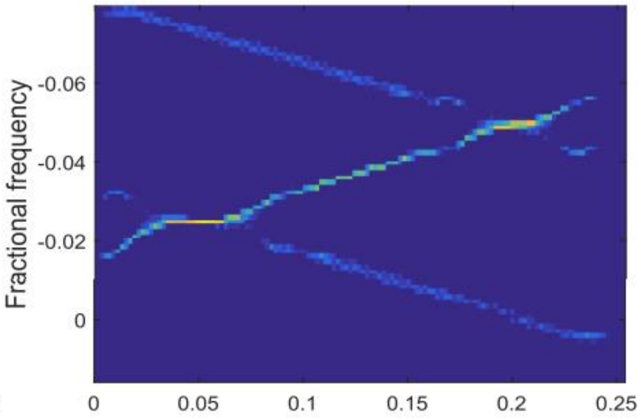
STFrFT α_2



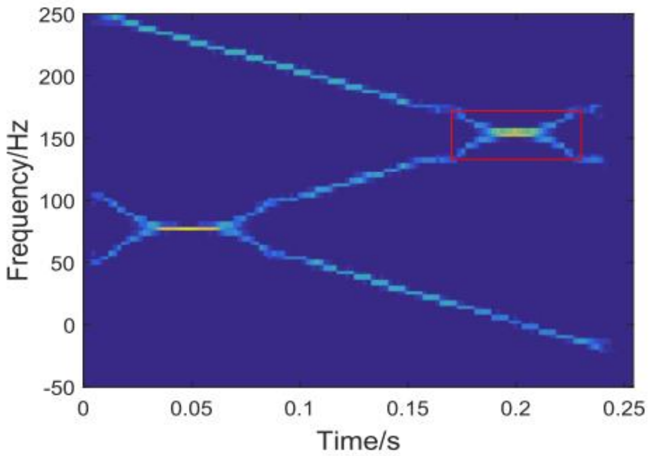
FrSST α_1



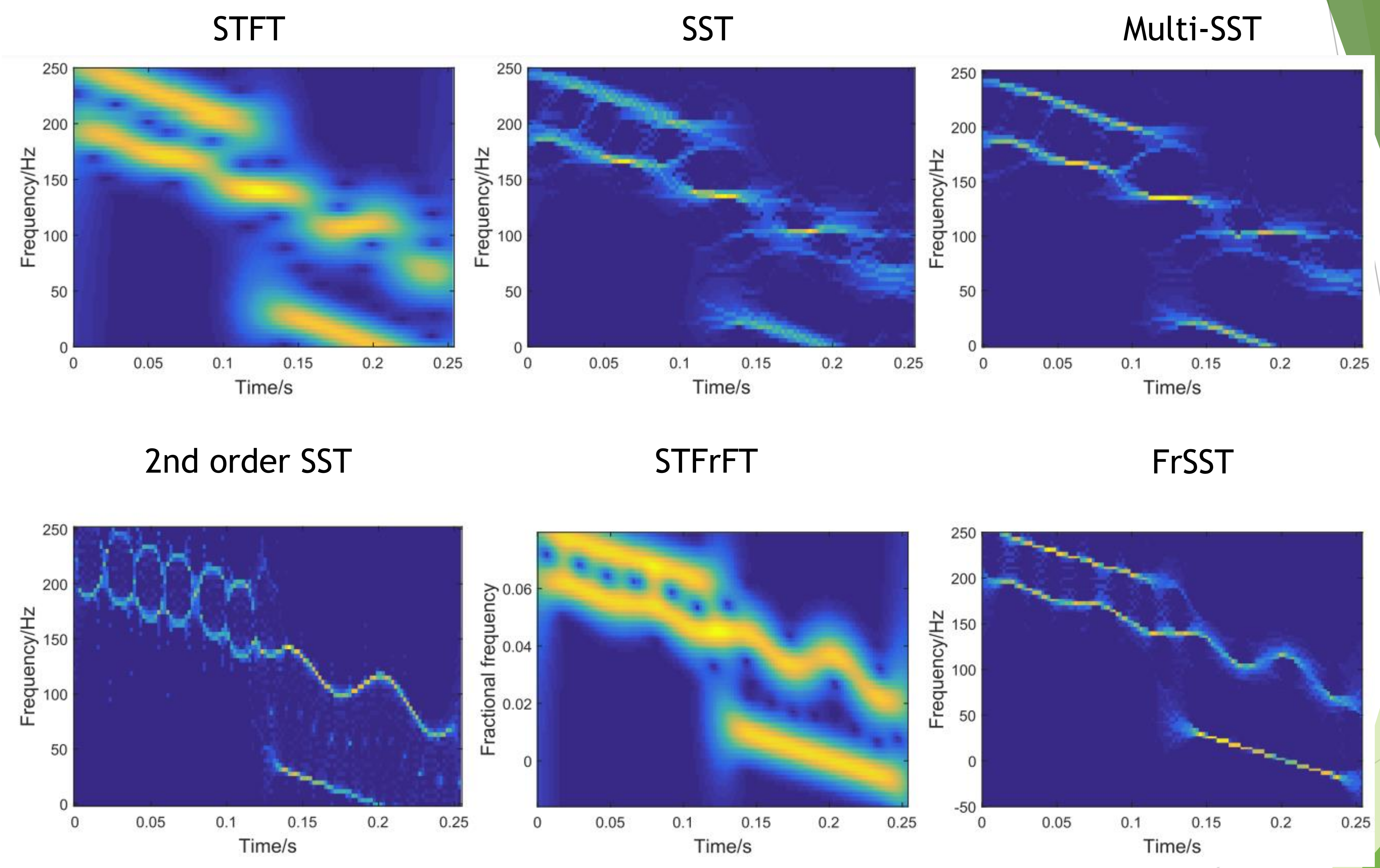
FrSST α_2



FrSST $\alpha_1 + \alpha_2$



Z. Zhao and G. Li, “Synchrosqueezing-based short-time fractional fourier transform,” IEEE Transactions on Signal Processing, vol. 71, pp. 279–294, 2023.



Z. Zhao and G. Li, “Synchrosqueezing-based short-time fractional fourier transform,” IEEE Transactions on Signal Processing, vol. 71, pp. 279–294, 2023.

Conclusion

- General model to multicomponent signal with 2-order phase.
- General SST to FrSST based on STFrFT.

Reference

1. Z. Zhao and G. Li, “Synchrosqueezing-based short-time fractional fourier transform,” *IEEE Transactions on Signal Processing*, vol. 71, pp. 279–294, 2023.